21. Areas of Bounded Regions

Exercise 21.1

1. Question

Using integration, find the area of the region bounded between the line x = 2 and the parabola $y^2 = 8x$.

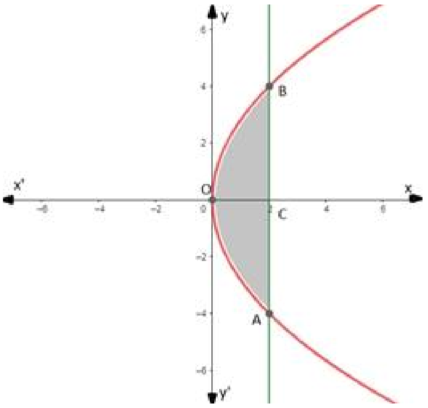
Answer

Given equations are:

x = 2 ...... (1)

And $y^2 = 8x$ ...... (2)

Equation (1) represents a line parallel to y - axis at a distance of 2 units and equation (2) represents a parabola with vertex at origin and x - axis as its axis, A rough sketch is given as below: -



We have to find the area of shaded region.

Required area

= shaded region OBAO

= 2 (shaded region OBCO) (as it is symmetrical about the x - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 2\int_0^2 y\,dx$ (As x is between (0,2) and the value of y varies)

$= 2\int_0^2 \sqrt{8x}\,dx$ (as $y^2 = 8x \Rightarrow y = \sqrt{8x}$)

$= 2\sqrt{8}\int_0^2 (x)^{\frac{1}{2}}dx$

On integrating we get,

$= 2\sqrt{8}\left[\frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^2 = 2\sqrt{8}\left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2$

$= 2\sqrt{8}\times\frac{2}{3}\left[(x)^{\frac{3}{2}}\right]_0^2$

On applying the limits, we get,

$= \frac{4\sqrt{8}}{3}\left[(2)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right] = \frac{4\sqrt{8}}{3}\times\sqrt{2^3}$

$= \frac{4\sqrt{8}}{3} \times \sqrt{8} = \frac{32}{3}$

Hence the area of the region bounded between the line x = 2 and the parabola $y^2 = 8x$ is equal to $\frac{32}{3}$ square units.

2. Question

Using integration, find the area of the region bounded by the line y – 1 = x, the x – axis and the ordinates x = – 2 and x = 3.

Answer

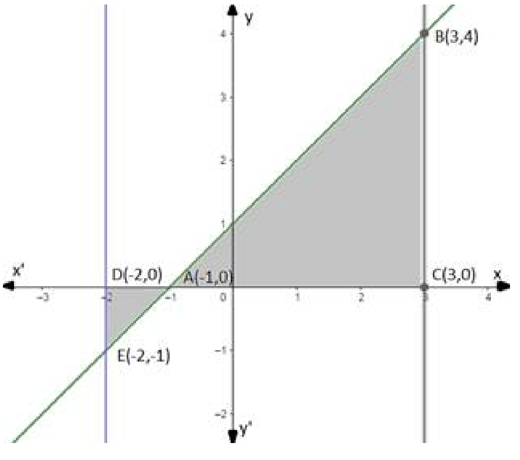
Given equations are:

y – 1 = x (is a line that meets at axes at (0,1) and (– 1,0))

x = – 2 (is line parallel to y – axis at a distance of 2 units to the left)

x = 3 (is line parallel to y - axis at a distance of 3 units to the right)

A rough sketch is given as below: -



We have to find the area bounded by these three lines with the x - axis, i.e., area of the shaded region.

Required area

= shaded region ABCA + shaded region ADEA

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_{-1}^{3} y\,dx + \left|\int_{-2}^{-1} y\,dx\right|$ (As x is between (– 1,3) for the region ABCA and it is between (– 2, – 1) for the region ADEA and the value of y varies)

$= \int_{-1}^{3} (x+1)dx + \left|\int_{-2}^{-1} (x+1)\,dx\right|$ (as y – 1 = x ⇒ y = x + 1)

$= \int_{-1}^{3} (x)dx + \int_{-1}^{3} (x^0)dx + \left|\int_{-2}^{-1} (x)\,dx + \int_{-2}^{-1} (x^0)\,dx\right|$ (as $x^0 = 1$)

On integrating we get,

$= \left[\frac{x^{1+1}}{1+1}\right]_{-1}^{3} + \left[\frac{x^{0+1}}{0+1}\right]_{-1}^{3} + \left|\left[\frac{x^{1+1}}{1+1}\right]_{-2}^{-1} + \left[\frac{x^{0+1}}{0+1}\right]_{-2}^{-1}\right|$

$= \left[\frac{x^2}{2} + \frac{x^1}{1}\right]_{-1}^{3} + \left|\left[\frac{x^2}{2} + \frac{x^1}{1}\right]_{-2}^{-1}\right|$ (Combining terms with same limits)

On applying the limits, we get

$$= \left(\frac{3^2}{2} + \frac{3^1}{1}\right) - \left(\frac{(-1)^2}{2} + \frac{(-1)^1}{1}\right) + \left|\left(\frac{(-1)^2}{2} + \frac{(-1)^1}{1}\right) - \left(\frac{(-2)^2}{2} + \frac{(-2)^1}{1}\right)\right|$$

$$= \left(\frac{9}{2} + 3\right) - \left(\frac{1}{2} - 1\right) + \left|\left(\frac{1}{2} - 1\right) - \left(\frac{4}{2} - 2\right)\right|$$

$$= \left(\frac{9}{2} + 3\right) + \frac{1}{2}\left|\left(-\frac{1}{2}\right) - (0)\right|$$

$$= \frac{9+6}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= \frac{15+1+1}{2} = \frac{17}{2}$$

Hence the area of the region bounded by the line y – 1 = x, the x – axis and the ordinates x = – 2 and x = 3 is equal to $\frac{17}{2}$ square units.

3. Question

Find the area the region bounded by the parabola $y^2 = 4ax$ and the line x = a.

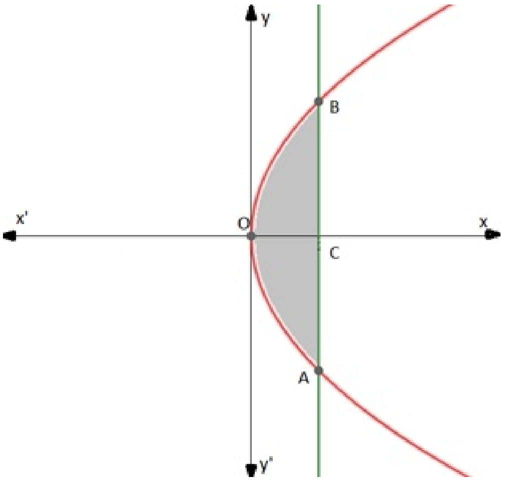
Answer

Given equations are:

x = a ...... (1)

And $y^2 = 4ax$ ...... (2)

Equation (1) represents a line parallel to the y - axis at a distance of units and equation (2) represents a parabola with vertex at origin and x - axis as its axis; A rough sketch is given as below: -



We have to find the area of the shaded region.

Required area

= shaded region OBAO

= 2 (shaded region OBCO) (as it is symmetrical about the x - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 2\int_0^a y\,dx$ (As x is between (0,a) and the value of y varies)

$= 2\int_0^a \sqrt{4ax}\,dx$ (as $y^2 = 4ax \Rightarrow y = \sqrt{4ax}$)

$$= 2\sqrt{4a}\int_0^a (x)^{\frac{1}{2}}\,dx$$

On integrating we get,

$$= 2\sqrt{4a}\left[\frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^a = 2\sqrt{4a}\left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$

$$= 2\sqrt{4a} \times \frac{2}{3}\left[(x)^{\frac{3}{2}}\right]_0^a$$

On applying the limits, we get,

$$= \frac{4\sqrt{4a}}{3}\left[(a)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right] = \frac{4\sqrt{4a}}{3} \times \sqrt{a^3}$$

$$= \frac{4 \times 2\sqrt{a}}{3} \times a\sqrt{a} = \frac{8a^2}{3}$$

Hence the area of the region bounded between the line x = a and the parabola $y^2 = 4ax$ is equal to $\frac{8a^2}{3}$ square units.

4. Question

Find the area lying above the x - axis and under the parabola $y = 4x - x^2$.

Answer

Given equations are:

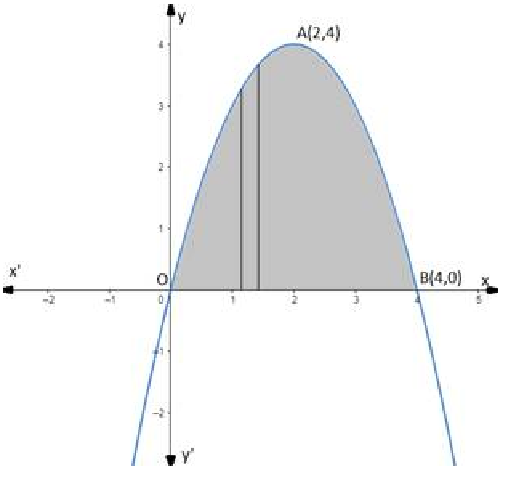
x – axis ...... (1)

And $y = 4x - x^2$ ...... (2)

$\Rightarrow y + 4 = -(x^2 - 4x - 4)$ (adding 4 on both sides)

$\Rightarrow -(y + 4) = (x - 2)^2$

equation (2) represents a downward parabola with vertex at (2,4) and passing through (0,0) and (4,0) on the x – axis, A rough sketch is given as below: –



We have to find the area of the shaded region.

Required area

= shaded region OABO (as it is symmetrical about the x - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_0^4 y\,dx$ (As x is between (0,4) and the value of y varies)

$= \int_0^4 (4x - x^2)\, dx$ (as y = 4x – x^2)

$= \int_0^4 (4x)\, dx - \int_0^4 (x^2)\, dx$

On integrating we get,

$= 4\left[\frac{(x)^{1+1}}{1+1}\right]_0^4 - \left[\frac{(x)^{2+1}}{2+1}\right]_0^4 = 4\left[\frac{(x)^2}{2}\right]_0^4 - \left[\frac{(x)^3}{3}\right]_0^4$

On applying the limits, we get,

$= 4\left[\frac{(4)^2}{2} - 0\right] - \left[\frac{(4)^3}{3} - 0\right]$

$= \frac{4 \times 16}{2} - \frac{64}{3} = \frac{64 \times 3 - 64 \times 2}{6}$

$= \frac{64}{2}\left(\frac{3-2}{3}\right) = \frac{32}{3}$

Hence the area lying above the x - axis and under the parabola y = 4x – x^2 is equal to $\frac{32}{3}$ square units.

5. Question

Draw a rough sketch to indicate the bounded between the curve y^2 = 4x and the line x = 3. Also, find the area of this region

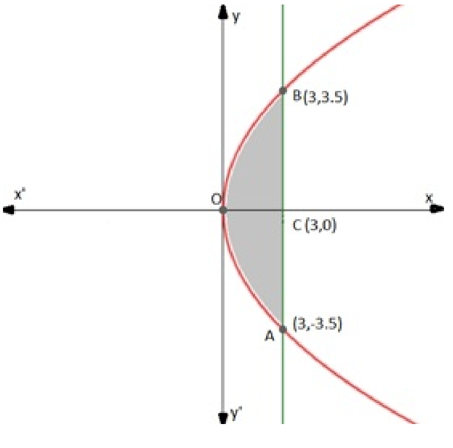
Answer

Given equations are:

x = 3 ...... (1)

And y^2 = 4x ...... (2)

Equation (1) represents a line parallel to the y - axis at a distance of 3 units and equation (2) represents a parabola with vertex at origin and x - axis as its axis; A rough sketch is given as below: -



We have to find the area of shaded region.

Required area

= shaded region OBCAO

= 2 (shaded region OBCO) (as it is symmetrical about the x - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 2\int_0^3 y\, dx$ (As x is between (0,3) and the value of y varies)

$= 2\int_0^3 \sqrt{4x}\,dx$ (as $y^2 = 4x \Rightarrow y = \sqrt{4x}$)

$$= 2\sqrt{4}\int_0^3 (x)^{\frac{1}{2}}dx$$

On integrating we get,

$$= 2 \times 2\left[\frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^3 = 4\left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$

$$= 4 \times \frac{2}{3}\left[(x)^{\frac{3}{2}}\right]_0^3$$

On applying the limits, we get,

$$= \frac{8}{3}\left[(3)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right] = \frac{8}{3} \times \sqrt{3^3}$$

$$= \frac{8}{3} \times 3\sqrt{3} = 8\sqrt{3}$$

Hence the area of the region bounded between the line x = 3 and the parabola $y^2 = 4x$ is equal to $8\sqrt{3}$ square units.

6. Question

Make a rough sketch of the graph of the function $y = 4 - x^2$, $0 \le x \le 2$ and determine the area enclosed by the curve, the x - axis and the lines x = 0 and x = 2.

Answer

Given equations are:

x – axis ...... (1)

x = 0 ...... (2)

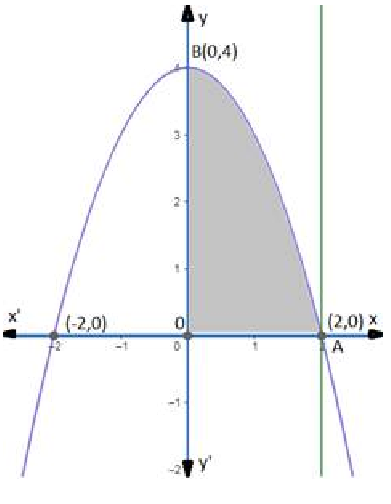
x = 2 ...... (3)

And $y = 4 - x^2$, $0 \le x \le 2$ ...... (4)

$\Rightarrow y = -(x^2 - 4) \Rightarrow x^2 = -(y - 4)$

equation (4) represents a downward parabola with vertex at (0,4) and passing through (2,0) and (– 2,0) on x – axis, equation (3) represents a line parallel to y – axis at a distance of 2 units and equation (2) represents y - axis.

A rough sketch is given as below: -



We have to find the area of the shaded region.

Required area

= shaded region OABO

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_0^2 y\,dx$ (As x is between (0,2) and the value of y varies)

$= \int_0^2 (4 - x^2)\,dx$ (as y = 4 – x^2)

$= \int_0^2 (4x^0)\,dx - \int_0^2 (x^2)\,dx$ (as x^0 = 1)

On integrating we get,

$$= 4\left[\frac{(x)^{0+1}}{0+1}\right]_0^2 - \left[\frac{(x)^{2+1}}{2+1}\right]_0^2 = 4\left[\frac{(x)^1}{1}\right]_0^2 - \left[\frac{(x)^3}{3}\right]_0^2$$

On applying the limits, we get,

$$= 4\left[\frac{(2)^1}{1} - 0\right] - \left[\frac{(2)^3}{3} - 0\right]$$

$$= 8 - \frac{8}{3} = \frac{8 \times 3 - 8}{3}$$

$$= \left(\frac{24 - 8}{3}\right) = \frac{16}{3}$$

Hence the area enclosed by the curve, the x - axis and the lines x = 0 and x = 2is equal to $\frac{16}{3}$ square units.

7. Question

Sketch the graph of $y = \sqrt{x+1}$ in [0,4] and determine the area of the region enclosed by the curve, the x - axis and the lines x = 0, x = 4

Answer

Given equations are:

x – axis ...... (1)

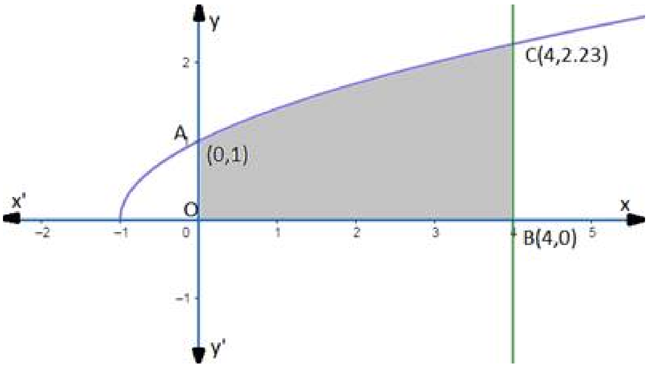
x = 0 ...... (2)

x = 4 ...... (3)

And $y = \sqrt{x+1}$, 0 ≤ x ≤ 4 ...... (4)

equation (4) represents a half parabola with vertex at (– 1,0) and passing through (4,0) on x – axis, equation (3) represents a line parallel to y - axis at a distance of 4 units and equation (2) represents y - axis.

A rough sketch is given as below: -



We have to find the area of the shaded region.

Required area

= shaded region AOBC

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_0^4 y\,dx$ (As x is between (0,4) and the value of y varies)

$= \int_0^4 \sqrt{x+1}\,dx$ (as $y = \sqrt{x+1}$)

$$= \int_0^4 (x+1)^{\frac{1}{2}}dx$$

Substitute $u = x + 1 \Rightarrow dx = du$

So the above equation becomes,

$$= \int_0^4 (u)^{\frac{1}{2}}du$$

On integrating we get,

$$= \left[\frac{(u)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^4 = \left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

On applying the limits we get,

$$= \left[\frac{2(4+1)^{\frac{3}{2}}}{3} - \frac{2(0+1)^{\frac{3}{2}}}{3}\right]$$

$$= \frac{2\sqrt{5^3}}{3} - \frac{2}{3} = \frac{2\sqrt{5^3}-2}{3}$$

$$= \frac{2(\sqrt{5^3}-1)}{3}$$

Hence the area of the region enclosed by the curve, the x - axis and the lines x = 0, x = 4is equal to $\frac{2(\sqrt{5^3}-1)}{3}$ square units.

8. Question

Find the area under the curve $y = \sqrt{6x+4}$ above x - axis from x = 0 to x = 2. Draw a sketch of curve also.

Answer

Given equations are:

x – axis ...... (1)

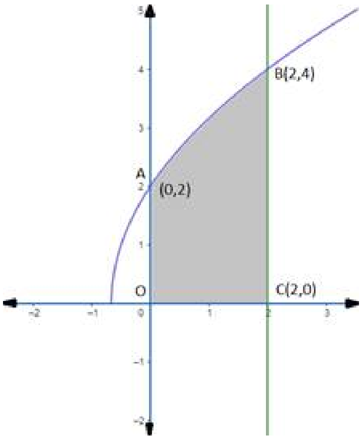
x = 0 ...... (2)

x = 2 ...... (3)

And $y=\sqrt{6x+4}$ ...... (4)

equation (4) represents a half parabola with vertex at $\left(-\frac{2}{3},0\right)$ and passing through (2,0) on x - axis, equation (3) represents a line parallel to y - axis at a distance of 2 units and equation (2) represents y - axis.

A rough sketch is given as below: -



We have to find the area of shaded region.

Required area

= shaded region OABC

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$=\int_0^2 y\,dx$ (As x is between (0,2) and the value of y varies)

$=\int_0^2 \sqrt{6x+4}\,dx$ (as $y=\sqrt{6x+4}$)

$$=\int_0^2 (6x+4)^{\frac{1}{2}}\,dx$$

Substitute $u=6x+4 \Rightarrow dx=\frac{1}{6}du$

So the above equation becomes,

$$=\frac{1}{6}\int_0^2 (u)^{\frac{1}{2}}\,du$$

On integrating we get,

$$=\frac{1}{6}\left[\frac{(u)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^2=\frac{1}{6}\left[\frac{(6x+4)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2$$

On applying the limits we get,

$$=\frac{1}{6}\left[\frac{2(6(2)+4)^{\frac{3}{2}}}{3}-\frac{2(6(0)+4)^{\frac{3}{2}}}{3}\right]$$

$$= \frac{(16)^{\frac{3}{2}}}{9} - \frac{(4)^{\frac{3}{2}}}{9} = \frac{\sqrt{(16)^3}}{9} - \frac{\sqrt{(4)^3}}{9}$$

$$= \frac{64}{9} - \frac{8}{9} = \frac{56}{9}$$

Hence the area under the curve $y = \sqrt{6x+4}$ above x - axis from x = 0 to x = 2 is equal to $\frac{56}{9}$ square units.

9. Question

Draw the rough sketch of $y^2 + 1 = x$, $x \le 2$. Find the area enclosed by the curve and the line x = 2

Answer

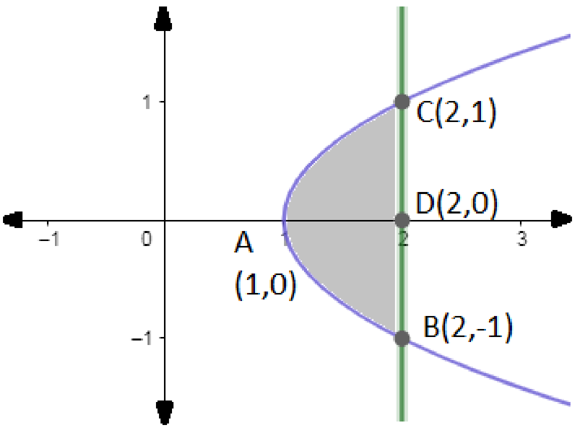
Given equations are:

x = 2 ...... (1)

And $y^2 + 1 = x$, $x \le 2$ ...... (2)

equation (2) represents a parabola with vertex at (1,0) and passing through (2,0) on x - axis, equation (1) represents a line parallel to y - axis at a distance of 2 units.

A rough sketch is given as below: -



We have to find the area of shaded region.

Required area

= shaded region ABCA

= 2 (shaded region ACDA) (as it is symmetrical about the x - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 2\int_1^2 y\,dx$ (As x is between (1,2) and the value of y varies)

$= 2\int_1^2 \sqrt{x-1}\,dx$ (as $y^2 + 1 = x$)

$$= 2\int_1^2 (x-1)^{\frac{1}{2}}dx$$

Substitute $u = x - 1 \Rightarrow dx = du$

So the above equation becomes,

$$= 2\int_1^2 (u)^{\frac{1}{2}}du$$

On integrating we get,

$$= 2\left[\frac{(u)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^2 = 2\left[\frac{(x-1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2$$

On applying the limits we get,

$$= \left[\frac{4(2-1)^{\frac{3}{2}}}{3} - \frac{4(1-1)^{\frac{3}{2}}}{3}\right]$$

$$= \frac{4\sqrt{1^3}}{3} - \frac{0}{3} = \frac{4}{3}$$

Hence the area enclosed by the curve and the line x = 2 is equal to $\frac{4}{3}$ square units.

10. Question

Draw a rough sketch of the graph of the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and evaluate the area of the region under the curve and above the x - axis

Answer

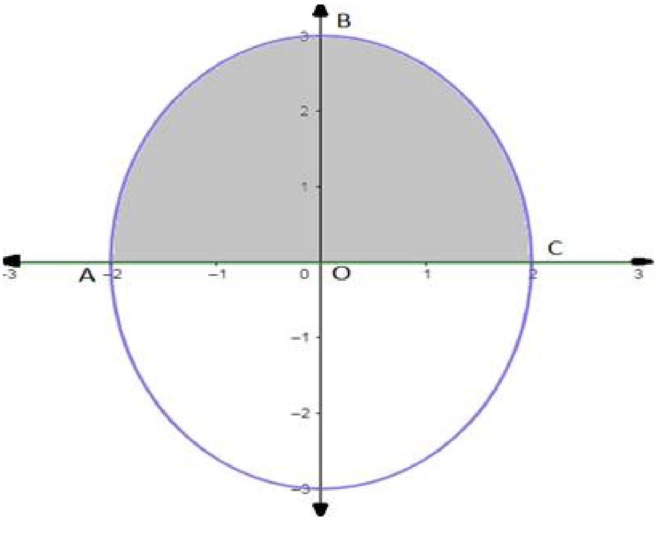
Given equations are:

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ ...... (1)

And x - axis ...... (2)

equation (1) represents an eclipse that is symmetrical about the x - axis and also about the y - axis, with center at origin and passes through (±2, 0) and (0, ±3).

A rough sketch is given as below: -



We have to find the area of shaded region.

Required area

= shaded region ABCA

= 2 (shaded region OBCO) (as it is symmetrical about the x - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 2\int_0^2 y\,dx$ (As x is between (0,2) and the value of y varies)

$= 2\int_0^2 3\sqrt{1-\frac{x^2}{4}}dx$ (as $\frac{x^2}{4}+\frac{y^2}{9}=1 \Rightarrow \frac{y^2}{9}=1-\frac{x^2}{4} \Rightarrow y=3\sqrt{1-\frac{x^2}{4}}$)

$$= 6\int_0^2 \frac{1}{2}\sqrt{4-x^2}\,dx$$

$$= 3\int_0^2 \sqrt{4-x^2}\,dx$$

Substitute $x = 2\sin u \Rightarrow u = \sin^{-1}\left(\frac{x}{2}\right), dx = 2\cos u\,du$

So the above equation becomes,

$$= 3\int_0^2 \sqrt{4-(2\sin u)^2}\,(2\cos u\,du)$$

$$= 3\int_0^2 2\cos u\sqrt{4-4\sin^2 u}\,du$$

We know, $4-4\sin^2 u = 4(1-\sin^2 u) = 4\cos^2 u$

So the above equation becomes,

$$= 3\int_0^2 2\cos u\sqrt{4\cos^2 u}\,du$$

$$= 3\int_0^2 4\cos^2 u\,du$$

Apply reduction formula:

$$\left[\int \cos^n u\,du = \frac{n-1}{n}\int \cos^{n-2}u\,du + \frac{(\cos^{n-1}u\sin u)}{n}\right]$$

On integrating we get,

$$= 12\left[\frac{1}{2}\int 1du + \frac{(\cos u\sin u)}{2}\right]_0^2$$

$$= 12\left[\frac{u}{2} + \frac{(\cos u\sin u)}{2}\right]_0^2 \left[\because \int 1du = u\right]$$

Undo the substituting, we get

$$= 12\left[\frac{\sin^{-1}\left(\frac{x}{2}\right)}{2} + \frac{\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\sin\left(\sin^{-1}\left(\frac{x}{2}\right)\right))}{2}\right]_0^2$$

$$= 12\left[\frac{\sin^{-1}\left(\frac{x}{2}\right)}{2} + \frac{\sqrt{1-\frac{x^2}{4}}\times\frac{x}{2}}{2}\right]_0^2$$

$$= 6\left[\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{1-\frac{x^2}{4}}\right]_0^2$$

On applying the limits we get,

$$= 6\left[\left(\sin^{-1}\left(\frac{2}{2}\right) + \frac{2}{2}\sqrt{1 - \frac{2^2}{4}}\right) - \left(\sin^{-1}\left(\frac{0}{2}\right) + \frac{0}{2}\sqrt{1 - \frac{0^2}{4}}\right)\right]$$

$$= 6\left[\left(\sin^{-1}(1) + \sqrt{1-1}\right) - 0\right]$$

$$= 6\left[\left(\frac{\pi}{2} + 0\right)\right] = 3\pi$$

Hence the area of the region under the given curve and above the x - axis is equal to 3π square units.

11. Question

Sketch the region $\{(x,y):9x^2 + 4y^2 = 36\}$ and the find the area of the region enclosed by it, using integration

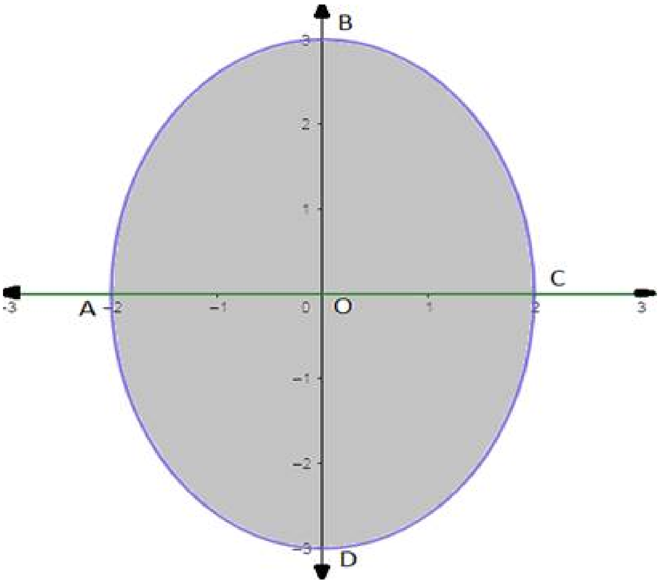
Answer

Given equation:

$9x^2 + 4y^2 = 36$ ...... (1)

equation (1) represents an eclipse that is symmetrical about the x - axis and also about the y - axis, with center at origin and passes through (±2, 0) and (0, ±3).

A rough sketch is given as below: -



We have to find the area of the shaded region.

Required area

= shaded region ABCDA

= 4 (shaded region OBCO) (as it is symmetrical about the x - axis as well as y - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 4\int_0^2 y\,dx$ (As x is between (0,2) and the value of y varies)

$= 4\int_0^2 3\sqrt{1 - \frac{x^2}{4}}\,dx$ (as $\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow \frac{y^2}{9} = 1 - \frac{x^2}{4} \Rightarrow y = 3\sqrt{1 - \frac{x^2}{4}}$)

$$= 12\int_0^2 \frac{1}{2}\sqrt{4 - x^2}\,dx$$

$$= 6\int_0^2 \sqrt{4 - x^2}\,dx$$

Substitute $x = 2\sin u \Rightarrow u = \sin^{-1}\left(\frac{x}{2}\right), dx = 2\cos u\, du$

So the above equation becomes,

$$= 6\int_0^2 \sqrt{4-(2\sin u)^2}\,(2\cos u\, du)$$

$$= 6\int_0^2 2\cos u\sqrt{4-4\sin^2 u}\, du$$

We know, $4-4\sin^2 u = 4(1-\sin^2 u) = 4\cos^2 u$

So the above equation becomes,

$$= 6\int_0^2 2\cos u\sqrt{4\cos^2 u}\, du$$

$$= 6\int_0^2 4\cos^2 u\, du$$

Apply reduction formula:

$$\left[\int \cos^n u\, du = \frac{n-1}{n}\int \cos^{n-2} u\, du + \frac{(\cos^{n-1} u \sin u)}{n}\right]$$

On integrating we get,

$$= 24\left[\frac{1}{2}\int 1du + \frac{(\cos u \sin u)}{2}\right]_0^2$$

$$= 24\left[\frac{u}{2} + \frac{(\cos u \sin u)}{2}\right]_0^2 \left[\because \int 1du = u\right]$$

Undo the substituting, we get

$$= 24\left[\frac{\sin^{-1}\left(\frac{x}{2}\right)}{2} + \frac{\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\sin\left(\sin^{-1}\left(\frac{x}{2}\right)\right)}{2}\right]_0^2$$

$$= 24\left[\frac{\sin^{-1}\left(\frac{x}{2}\right)}{2} + \frac{\sqrt{1-\frac{x^2}{4}}\times\frac{x}{2}}{2}\right]_0^2$$

$$= 12\left[\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{1-\frac{x^2}{4}}\right]_0^2$$

On applying the limits we get,

$$= 12\left[\left(\sin^{-1}\left(\frac{2}{2}\right) + \frac{2}{2}\sqrt{1-\frac{2^2}{4}}\right) - \left(\sin^{-1}\left(\frac{0}{2}\right) + \frac{0}{2}\sqrt{1-\frac{0^2}{4}}\right)\right]$$

$$= 12\left[\left(\sin^{-1}(1) + \sqrt{1-1}\right) - 0\right]$$

$$= 12\left[\left(\frac{\pi}{2} + 0\right)\right] = 6\pi$$

Hence the area of the region enclosed by it is equal to 6π square units.

12. Question

Draw a rough sketch of the graph of the function y=2√1–x²,x[0,1] and evaluate the are enclosed between the curve and the x–axis.

Answer

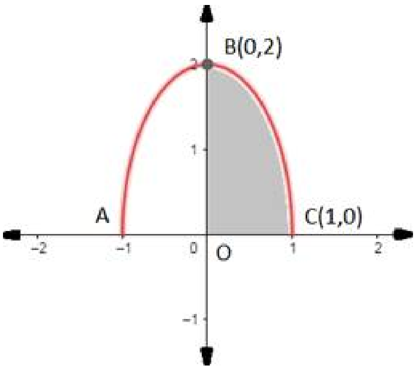
Given equation:

$y = 2\sqrt{1-x^2}, x \in [0,1]$ ...... (1)

$\Rightarrow \frac{y^2}{4} = 1 - x^2 \Rightarrow \frac{x^2}{1} + \frac{y^2}{4} = 1$

equation (1) represents a half eclipse that is symmetrical about the x - axis and also about the y - axis with center at origin and passes through (±1, 0) and (0, ±2). And x∈[0,1] is represented by region between y - axis and line x = 1.

A rough sketch is given as below: -



We have to find the area of shaded region.

Required area

= (shaded region OBCO)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_0^1 y\,dx$ (As x is between (0,1) and the value of y varies)

$= \int_0^1 2\sqrt{1-x^2}\,dx$ (as $y = 2\sqrt{1-x^2}$)

$= 2\int_0^1 \sqrt{1-x^2}\,dx$

Substitute $x = \sin u \Rightarrow u = \sin^{-1}(x), dx = \cos u\,du$

So the above equation becomes,

$= 2\int_0^1 \sqrt{1-(\sin u)^2}\,(\cos u\,du)$

$= 2\int_0^1 \cos u\sqrt{1-\sin^2 u}\,du$

We know, $1 - \sin^2 u = (1 - \sin^2 u) = \cos^2 u$

So the above equation becomes,

$= 2\int_0^1 \cos u\sqrt{\cos^2 u}\,du$

$= 2\int_0^1 \cos^2 u\, du$

Apply reduction formula:

$$\left[\int \cos^n u\, du = \frac{n-1}{n}\int \cos^{n-2} u\, du + \frac{(\cos^{n-1} u \sin u)}{n}\right]$$

On integrating we get,

$$= 2\left[\frac{1}{2}\int 1du + \frac{(\cos u \sin u)}{2}\right]_0^1$$

$$= 2\left[\frac{u}{2} + \frac{(\cos u \sin u)}{2}\right]_0^1 \left[\because \int 1du = u\right]$$

Undo the substituting, we get

$$= 2\left[\frac{\sin^{-1}(x)}{2} + \frac{\cos(\sin^{-1}(x))\sin(\sin^{-1}(x))}{2}\right]_0^1$$

$$= 2\left[\frac{\sin^{-1}(x)}{2} + \frac{\sqrt{1-x^2} \times x}{2}\right]_0^1$$

$$= \left[\sin^{-1}(x) + x\sqrt{1-x^2}\right]_0^1$$

On applying the limits we get,

$$= \left[\left(\sin^{-1}(1) + \sqrt{1-1}\right) - \left(\sin^{-1}(0) + 0\sqrt{1-0}\right)\right]$$

$$= \left[\left(\sin^{-1}(1) + \sqrt{1-1}\right) - 0\right]$$

$$= \left[\left(\frac{\pi}{2} + 0\right)\right] = \frac{\pi}{2}$$

Hence the area enclosed between the curve and the x - axis is equal to $\frac{\pi}{2}$ square units.

13. Question

Determine the area under the $y = \sqrt{a^2 - x^2}$ included between the lines x = 0 and x = 1

Answer

Given equations are :

$y = \sqrt{a^2 - x^2}$ ...... (1)

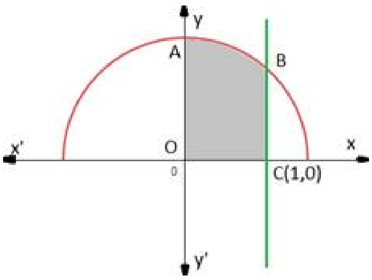
$\Rightarrow y^2 = a^2 - x^2 \Rightarrow x^2 + y^2 = a^2$

x = 0 (y – axis)

x = 1 (represents a line parallel to y - axis at a distance 1 to the right)

equation (1) represents a half eclipse that is symmetrical about the x - axis and also about the y - axis with center at origin.

A rough sketch is given as below: -



We have to find the area of shaded region.

Required area

= (shaded region OABCO)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_0^1 y\,dx$ (As x is between (0,1) and the value of y varies)

$= \int_0^1 \sqrt{a^2 - x^2}\,dx$ (as $y = \sqrt{a^2 - x^2}$)

$$= \int_0^1 \sqrt{a^2 - x^2}\,dx$$

Substitute $x = a\sin u \Rightarrow u = \sin^{-1}\left(\frac{x}{a}\right), dx = a\cos u\,du$

So the above equation becomes,

$$= \int_0^1 \sqrt{a^2 - (a\ \sin u)^2}\,(a\cos u\,du)$$

$$= \int_0^1 a\cos u\sqrt{a^2 - a^2\sin^2 u}\,du$$

We know, $a^2 - a^2\sin^2 u = a^2(1 - \sin^2 u) = a^2\cos^2 u$

So the above equation becomes,

$$= \int_0^1 a\cos u\sqrt{a^2\cos^2 u}\,du$$

$$= \int_0^1 a^2\ \cos^2 u\,du$$

Apply reduction formula:

$$\left[\int \cos^n u\,du = \frac{n-1}{n}\int \cos^{n-2} u\,du + \frac{(\cos^{n-1} u \sin u)}{n}\right]$$

On integrating we get,

$$= a^2\left[\frac{1}{2}\int 1du + \frac{(\cos u \sin u)}{2}\right]_0^1$$

$$= a^2\left[\frac{u}{2} + \frac{(\cos u \sin u)}{2}\right]_0^1 \left[\because \int 1du = u\right]$$

Undo the substituting, we get

$$= a^2\left[\frac{\sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}{2}\right]_0^1$$

$$= a^2\left[\frac{\sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{\sqrt{1-\frac{x^2}{a^2}}\times\frac{x}{a}}{2}\right]_0^1$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]_0^1$$

On applying the limits we get,

$$= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{1}{a}\right) + \frac{1}{a}\sqrt{1-\frac{1}{a^2}}\right) - \left(\sin^{-1}(0) + 0\sqrt{1-0}\right)\right]$$

$$= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{1}{a}\right) + \frac{1}{a}\sqrt{\frac{(a^2-1)}{a^2}}\right) - 0\right]$$

$$= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{1}{a}\right) + \frac{1}{a^2}\sqrt{(a^2-1)}\right)\right]$$

Hence the area under the $y = \sqrt{a^2 - x^2}$ included between the lines x = 0 and x = 1is equal to $\frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{1}{a}\right) + \frac{1}{a^2}\sqrt{(a^2-1)}\right)\right]$ square units.

14. Question

Using integration, find the area of the region bounded by the line 2y = 5x + 7, x - axis the lines x = 2 and x = 8.

Answer

Given equations are:

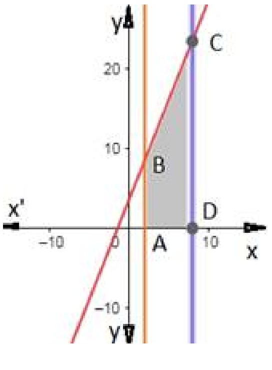
2y = 5x + 7 ...... (1)

x = 2 ...... (2)

x = 8 ...... (3)

Equation (1) represents line passing through $\left(-\frac{7}{5}, 0\right)$ and $\left(0, \frac{7}{2}\right)$. Equation (2), (3) shows line parallel to y - axis passing through (2,0), (8,0) respectively.

A rough sketch of curves is as below:



We have to find the area of shaded region.

Required area

= (shaded region ABCDA)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_2^8 y dx$ (As x is between (2,8) and the value of y varies)

$= \int_2^8 \left(\frac{5x+7}{2}\right) dx$ (as $2y = 5x + 7 \Rightarrow y = \frac{5x+7}{2}$)

$$= \frac{1}{2}\int_2^8 (5x+7)dx$$

$$= \frac{1}{2}\left[\int_2^8 (5x)dx + \int_2^8 (7)dx\right]$$

Now integrating by applying power rule, we get

$$= \frac{1}{2}\left[5\left[\frac{x^2}{2}\right]_2^8 + 7\left[\frac{x^1}{1}\right]_2^8\right]$$

Now applying the limits we get

$$= \frac{1}{2}\left[5\left[\frac{8^2}{2} - \frac{2^2}{2}\right] + 7[8-2]\right]$$

$$= \frac{1}{2}\left[5\left[\frac{64-4}{2}\right] + 42\right]$$

$$= \frac{1}{2}[5[30] + 42] = 96$$

Hence the area of the region bounded by the line 2y = 5x + 7, x - axis the lines x = 2 and x = 8 is equal to 96 square units.

15. Question

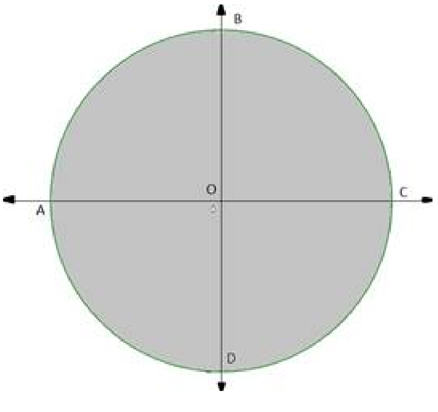
Using definite integrals, find the area of circle $x^2 + y^2 = a^2$

Answer

Given equations are :

$x^2 + y^2 = a^2$ ...... (1)

Equation (1) represents a circle with centre (0,0) and radius a, so it meets the axes at (±a,0), (0,±a). A rough sketch of the curve is given below: -



We have to find the area of shaded region.

Required area

= (shaded region ABCDA)

= 4(shaded region OBCO) (as the circle is symmetrical about the x - axis as well as the y - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 4\int_0^a y\,dx$ (As x is between (0,a) and the value of y varies)

$= 4\int_0^a \sqrt{a^2 - x^2}\,dx$ (as $y = \sqrt{a^2 - x^2}$)

Substitute $x = a\sin u \Rightarrow u = \sin^{-1}\left(\frac{x}{a}\right), dx = a\cos u\,du$

So the above equation becomes,

$$= 4\int_0^a \sqrt{a^2 - (a\sin u)^2}\,(a\cos u\,du)$$

$$= 4\int_0^a a\cos u\sqrt{a^2 - a^2\sin^2 u}\,du$$

We know, $a^2 - a^2\sin^2 u = a^2(1 - \sin^2 u) = a^2\cos^2 u$

So the above equation becomes,

$$= 4\int_0^a a\cos u\sqrt{a^2\cos^2 u}\,du$$

$$= 4\int_0^a a^2\cos^2 u\,du$$

Apply reduction formula:

$$\left[\int \cos^n u\,du = \frac{n-1}{n}\int \cos^{n-2} u\,du + \frac{(\cos^{n-1} u\sin u)}{n}\right]$$

On integrating we get,

$$= 4a^2\left[\frac{1}{2}\int 1du + \frac{(\cos u\sin u)}{2}\right]_0^a$$

$$= 4a^2\left[\frac{u}{2} + \frac{(\cos u\sin u)}{2}\right]_0^a \left[\because \int 1du = u\right]$$

Undo the substituting, we get

$$= 4a^2\left[\frac{\sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right))}{2}\right]_0^a$$

$$= 4a^2\left[\frac{\sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{\sqrt{1 - \frac{x^2}{a^2}} \times \frac{x}{a}}{2}\right]_0^a$$

$$= 2a^2\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\right]_0^a$$

On applying the limits we get,

$$= 2a^2\left[\left(\sin^{-1}\left(\frac{a}{a}\right) + \frac{a}{a}\sqrt{1 - \frac{a^2}{a^2}}\right) - \left(\sin^{-1}(0) + 0\sqrt{1-0}\right)\right]$$

$$= 2a^2\left[\left(\sin^{-1}(1) + 1\sqrt{0}\right) - 0\right]$$

$$= 2a^2\left(\frac{\pi}{2}\right) = \pi a^2$$

Hence the area of circle $x^2 + y^2 = a^2$ is equal to πa^2 square units.

16. Question

Using integration, find the area of the region bounded by the following curves, after making a rough sketch: y = 1 + |x + 1|, x = - 2, x = 3, y = 0.

Answer

Given equations are:

y = 1 + |x + 1|

y = 1 + x + 1, if x + 1≥ 0

y_2 = 2 + x …… (1), if x ≥ - 1

And y = 1 – (x + 1), if x + 1 < 0

y = 1 – x – 2, if x < - 1

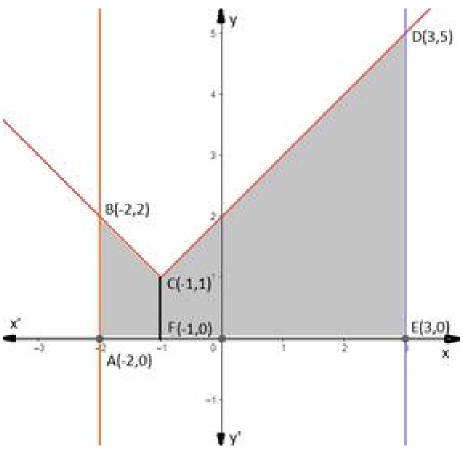
y_1 = - x …… (2), if x < - 1

x = - 2 …… (3)

x = 3 …… (4)

y = 0 …… (5)

So, equation (1) is straight line that passes thorough (0,2) and (- 1,1). Equation (2) is a line passing through (- 1,1) and (- 2,2) and it is enclosed by line x = - 2 and x = 3 which are lines parallel to y – axis and pass through (- 2,0) and (3,0) respectively, y = 0 is x – axis. So, a rough sketch of the curves is gives as: -



We have to find the area of shaded region.

Required area

= (shaded region ABCDEA)

= shaded region ABCFA + Shaded region FCDEFC

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_{-2}^{-1} y_1 dx + \int_{-1}^{3} y_2 dx$ (As x is between (- 2, - 1) in first shaded region and x is between (- 1,3) for the second shaded region)

$= \int_{-2}^{-1}(-x)dx + \int_{-1}^{3}(x+2)dx$ (from equation (1) and (2))

$$= -\int_{-2}^{-1}(x)dx + \int_{-1}^{3}(x)dx + \int_{-1}^{3}(2)dx$$

Now integrating by applying power rule, we get

$$= -\left[\frac{x^2}{2}\right]_{-2}^{-1} + \left[\frac{x^2}{2}\right]_{-1}^{3} + 2\left[\frac{x^1}{1}\right]_{-1}^{3}$$

Now applying the limits we get

$$= -\left[\frac{(-1)^2}{2} - \frac{(-2)^2}{2}\right] + \left[\frac{3^2}{2} - \frac{(-1)^2}{2}\right] + 2(3-(-1))$$

$$= -\left[\frac{1-4}{2}\right] + \left[\frac{9-1}{2}\right] + 8$$

$$= \frac{3}{2} + \frac{8}{2} + 8 = \frac{3+8+8\times 2}{2} = \frac{27}{2}$$

Hence the area of the region bounded by the curves, y = 1 + |x + 1|, x = - 2, x = 3, y = 0 is equal to $\frac{27}{2}$ square units.

17. Question

Sketch the graph y = |x - 5|. Evaluate $\int_0^1 |x-5|\,dx$. What does this value of the integral represent on the graph?

Answer

Given equations are:

y = |x - 5|

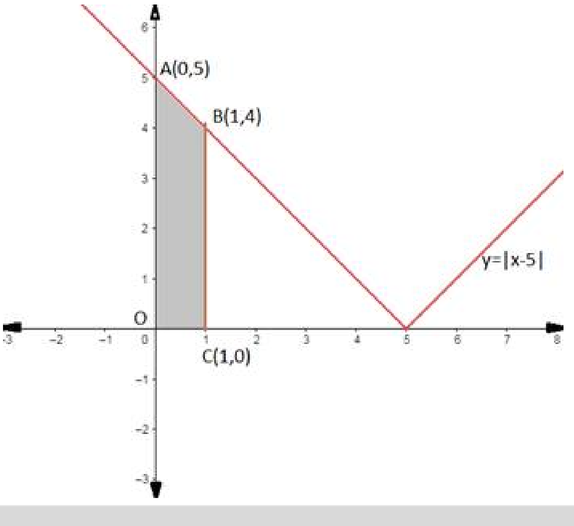
$y_1 = x - 5$, if $x - 5 \geq 0$

$y_1 = x - 5$ ...... (1), if $x \geq 5$

And $y_2 = -(x - 5)$, if $x - 5 < 0$

$y_2 = -(x - 5)$ ...... (2), if $x < 5$

So, equation (1) is straight line that passes thorough (5,0). Equation (2) is a line passing through (5,0) and (0,5). So, the graph of which is as follows:



$$\int_0^1 |x-5|dx$$

$= \int_0^1 y_2 dx$ (As when x is between (0,1) the given equation becomes y = - (x - 5) as shown in equation (2) shown ass shaded region in the above graph)

$= \int_0^1 -(x-5)dx$ (from equation (2))

$$= -\int_0^1 (x)dx + \int_0^1 (5)dx$$

Now integrating by applying power rule, we get

$$= -\left[\frac{x^2}{2}\right]_0^1 + 5\left[\frac{x^1}{1}\right]_0^1$$

Now applying the limits we get

$$= -\left[\frac{(1)^2}{2} - \frac{(0)^2}{2}\right] + 5(1-(0))$$

$$= -\left[\frac{1}{2}\right] + 5$$

$$= \frac{5 \times 2 - 1}{2} = \frac{9}{2}$$

Hence the value of $\int_0^1 |x-5|dx$ represents the area of the shaded region OABC (as shown in the graph) and is equal to $\frac{9}{2}$ square units.

18. Question

Sketch the graph y = |x + 3|. Evaluate $\int_{-6}^{0} |x+3|\, dx$. What does this integral represent on the graph?

Answer

Given equations are:

y = |x + 3|

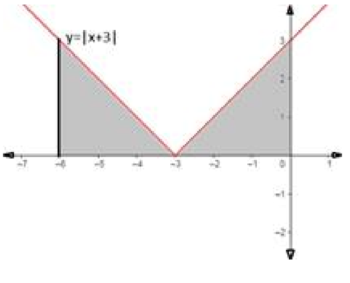
y_1 = x + 3, if x + 3≥ 0

y_1 = x + 3 ...... (1), if x ≥ - 3

And y_2 = - (x + 3), if x + 3 < 0

y_2 = - (x + 3) ...... (2), if x < - 3

So, equation (1) is straight line that passes thorough (- 3,0) and (0,3). Equation (2) is a line passing through (- 3,0). So, the graph of which is as follows:



$\int_{-6}^{0} |x+3| dx$

$= \int_{-6}^{-3} y_2 \, dx + \int_{-3}^{0} y_1 \, dx$ (As x is between (- 6, - 3) in first shaded region equation becomes as y_2 and when x is between (- 3,0) for the second shaded region equation becomes y_1)

$= \int_{-6}^{-3} -(x+3) dx + \int_{-3}^{0} (x+3) dx$ (from equation (2))

$$= -\int_{-6}^{-3} (x) dx - \int_{-6}^{-3} (3) dx + \int_{-3}^{0} (x) dx + \int_{-3}^{0} (3) dx$$

Now integrating by applying power rule, we get

$$= -\left[\frac{x^2}{2}\right]_{-6}^{-3} - 3\left[\frac{x^1}{1}\right]_{-6}^{-3} + \left[\frac{x^2}{2}\right]_{-3}^{0} + 3\left[\frac{x^1}{1}\right]_{-3}^{0}$$

Now applying the limits we get

$$= -\left[\frac{(-3)^2}{2} - \frac{(-6)^2}{2}\right] - 3\left[\frac{(-3)^1}{1} - \frac{(-6)^1}{1}\right] + \left[\frac{0^2}{2} - \frac{(-3)^2}{2}\right] + 3\left[\frac{0^1}{1} - \frac{(-3)^1}{1}\right]$$

$$= -\left[\frac{9-36}{2}\right] - 3[-3+6] + \left[-\frac{9}{2}\right] + 3[3]$$

$$= \left[\frac{27}{2}\right] - 9 - \left[\frac{9}{2}\right] + 9 = \left[\frac{27-9}{2}\right] = 9$$

Hence the value of $\int_{-6}^{0} |x+3| dx$ represents the area of the shaded region (as shown in the graph) and is equal to 9 square units.

19. Question

Sketch the graph y = |x + 1|. Evaluate $\int_{-4}^{2} |x+1| \, dx$. What does the value of this integral represent on the graph?

Answer

Given equations are:

y = |x + 1|

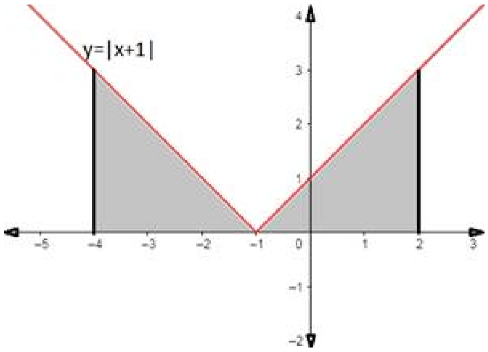
y_1 = x + 1, if x + 1 ≥ 0

y_1 = x + 1 ...... (1), if x ≥ - 1

And y_2 = - (x + 1), if x + 1 < 0

y_2 = - (x + 1) ...... (2), if x < - 1

So, equation (1) is straight line that passes thorough (- 1,0) and (0,1). Equation (2) is a line passing through (- 1,0). So, the graph of which is as follows:



$$\int_{-4}^{2}|x+1|dx$$

$= \int_{-4}^{-1} y_2\,dx + \int_{-1}^{2} y_1\,dx$ (As x is between (- 4, - 1) in first shaded region equation becomes as y_2 and when x is between (- 1,2) for the second shaded region equation becomes y_1)

$= \int_{-4}^{-1} -(x+1)dx + \int_{-1}^{2}(x+1)dx$ (from equation (2))

$$= -\int_{-4}^{-1}(x)dx - \int_{-4}^{-1}(1)dx + \int_{-1}^{2}(x)dx + \int_{-1}^{2}(1)dx$$

Now integrating by applying power rule, we get

$$= -\left[\frac{x^2}{2}\right]_{-4}^{-1} - \left[\frac{x^1}{1}\right]_{-4}^{-1} + \left[\frac{x^2}{2}\right]_{-1}^{2} + \left[\frac{x^1}{1}\right]_{-1}^{2}$$

Now applying the limits we get

$$= -\left[\frac{(-1)^2}{2} - \frac{(-4)^2}{2}\right] - \left[\frac{(-1)^1}{1} - \frac{(-4)^1}{1}\right] + \left[\frac{2^2}{2} - \frac{(-1)^2}{2}\right] + \left[\frac{2^1}{1} - \frac{(-1)^1}{1}\right]$$

$$= -\left[\frac{1-16}{2}\right] - [-1+4] + \left[\frac{4-1}{2}\right] + [2+1]$$

$$= \left[\frac{15}{2}\right] - 3 + \left[\frac{3}{2}\right] + 3 = \left[\frac{15+3}{2}\right] = 9$$

Hence the value of $\int_{-4}^{2}|x+1|dx$ represents the area of the shaded region (as shown in the graph) and is equal to 9 square units.

20. Question

Draw a rough sketch of the curve xy –3x – 2y – 10 = 0, x - axis and the lines x = 3, x = 4.

Answer

Given equations are:

xy –3x – 2y – 10 = 0 …..(i)

y (x - 2) = 3x + 10

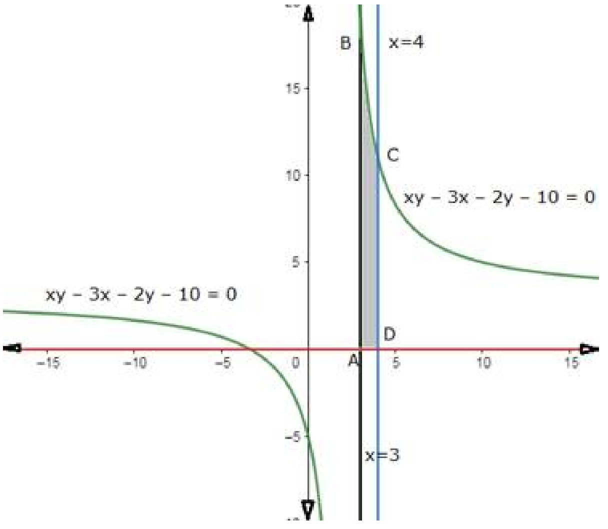
$\Rightarrow y = \frac{3x+10}{x-2}$ …..(ii)

x - axis …..(iii)

x = 3 ……(iv)

x = 4 …..(v)

A rough sketch of the curves is given below: -



We have to find the area of shaded region.

Required area

= (shaded region ABCDA)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_3^4 y\mathrm{d}x$ (As x is between (3,4) and the value of y varies)

$= \int_3^4 \left(\frac{3x+10}{x-2}\right)\mathrm{d}x$ (from equation(ii))

Substitute u = x−2 ⟶ dx = du

$$= \int_3^4 \left(\frac{3(u+2)+10}{u}\right)du$$

$$= \int_3^4 \left(\frac{3u+16}{u}\right)du$$

$$= \int_3^4 \left(3+\frac{16}{u}\right)du$$

$$= \int_3^4 (3)du + \int_3^4 \left(\frac{16}{u}\right)dx$$

Now on integrating we get

$= [3u + 16\log u]_3^4$

Undo substitution, we get

$= [3(x-2) + 16\log(x-2)]_3^4$

$= [3(x-2) + 16\log(x-2)]_3^4$

On applying the limits we get

$= [3(4-2) + 16\ \log(4-2)] - [3(3-2) + 16\ \log(3-2)]$

$= [6 + 16\ \log(2)] - [3 + 16\ \log(1)]$

$= 3 + 16(\log 2)$

Hence the area of the region bounded by the curves, xy –3x – 2y – 10 = 0, x - axis and the lines x = 3, x = 4 is equal to $3 + 16(\log 2)$ square units.

21. Question

Draw a rough sketch of the curve $y = \frac{\pi}{2} + 2\sin^2 x$ and find the area between x - axis, the curve and the ordinates x = 0, x = π.

Answer

Given equations are:

$y = \frac{\pi}{2} + 2\sin^2 x$ .....(i)

x - axis .....(ii)

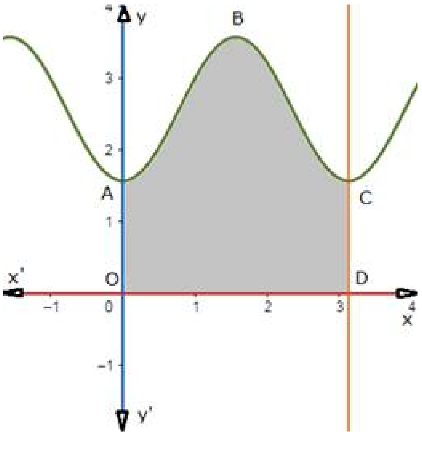
x = 0 ......(iii)

x = π .....(iv)

A table for values of $y = \frac{\pi}{2} + 2\sin^2 x$ is: -

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}, \frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}, \frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\frac{\pi}{2} + 2\sin^2 x$	1.57	2.07	2.57 3.07	3.57	3.07 2.57	2.07	1.57

A rough sketch of the curves is given below: -



We have to find the area of shaded region.

Required area

= (shaded region ABCDOA)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_0^\pi y dx$ (As x is between (0, π) and the value of y varies)

$= \int_0^\pi \left(\frac{\pi}{2} + 2\sin^2 x\right) dx$ (as $y = \frac{\pi}{2} + 2\sin^2 x$)

$= \int_0^\pi \left(\frac{\pi}{2}\right) dx + 2\int_0^\pi (\sin^2 x) dx$

Apply reduction formula:

$$\left[\int \sin^n x\,dx = \frac{n-1}{n}\int \sin^{n-2} x\,dx - \frac{(\sin^{n-1} x \cos x)}{n}\right]$$

On integrating we get,

$= \frac{\pi}{2}(x)_0^{\pi} + 2\left[\frac{1}{2}\int_0^{\pi}(1)dx - \frac{\sin x \cos x}{2}\right]_0^{\pi}$

$= \frac{\pi}{2}(x)_0^{\pi} + [x - \sin x \cos x]_0^{\pi}$

On applying the limits we get

$= \frac{\pi}{2}(\pi - 0) + \pi - \sin(\pi)\cos(\pi) - (0 - \sin(0)\cos(0))$

$= \frac{\pi^2}{2} + \pi - (1)(0) = \frac{\pi^2}{2} + \pi$

Hence the area between x - axis, the curve and the ordinates x = 0, x =π is equal to $\frac{\pi^2}{2} + \pi$ square units.

22. Question

Draw a rough sketch of the curve $y = \frac{\pi}{2} + 2\sin^2 x$ and find the area between x - axis, the curve and the ordinates x = 0, x = π.

Answer

Given equations are:

$y = \frac{\pi}{2} + 2\sin^2 x$ .....(i)

x - axis .....(ii)

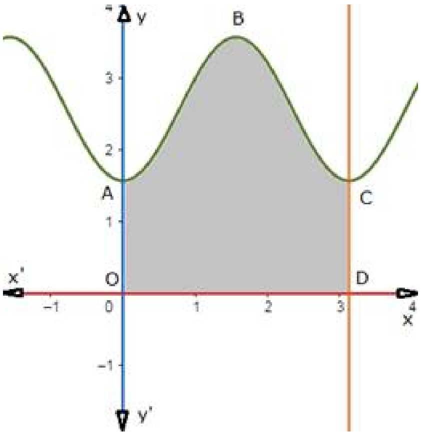
x = 0 ......(iii)

x = π .....(iv)

A table for values of $y = \frac{\pi}{2} + 2\sin^2 x$ is: -

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}, \frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}, \frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\frac{\pi}{2} + 2\sin^2 x$	1.57	2.07	2.57 3.07	3.57	3.07 2.57	2.07	1.57

A rough sketch of the curves is given below: -



We have to find the area of shaded region.

Required area

= (shaded region ABCDOA)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_0^\pi y dx$ (As x is between (0, π) and the value of y varies)

$= \int_0^\pi \left(\frac{\pi}{2} + 2\sin^2 x\right) dx$ (as $y = \frac{\pi}{2} + 2\sin^2 x$)

$$= \int_0^\pi \left(\frac{\pi}{2}\right) dx + 2\int_0^\pi (\sin^2 x) dx$$

Apply reduction formula:

$$\left[\int \sin^n x\, dx = \frac{n-1}{n}\int \sin^{n-2} x\, dx - \frac{(\sin^{n-1} x \cos x)}{n}\right]$$

On integrating we get,

$$= \frac{\pi}{2}(x)_0^\pi + 2\left[\frac{1}{2}\int_0^\pi (1) dx - \frac{\sin x \cos x}{2}\right]_0^\pi$$

$$= \frac{\pi}{2}(x)_0^\pi + [x - \sin x \cos x]_0^\pi$$

On applying the limits we get

$$= \frac{\pi}{2}(\pi - 0) + \pi - \sin(\pi)\cos(\pi) - (0 - \sin(0)\cos(0))$$

$$= \frac{\pi^2}{2} + \pi - (1)(0) = \frac{\pi^2}{2} + \pi$$

Hence the area between x - axis, the curve and the ordinates x = 0, x =π is equal to $\frac{\pi^2}{2} + \pi$ square units.

23. Question

Find the area bounded by the curve y = cosx, x - axis and the ordinates x = 0 and x = 2π.

Answer

Given equations are:

y = cos x .....(i)

x - axis .....(ii)

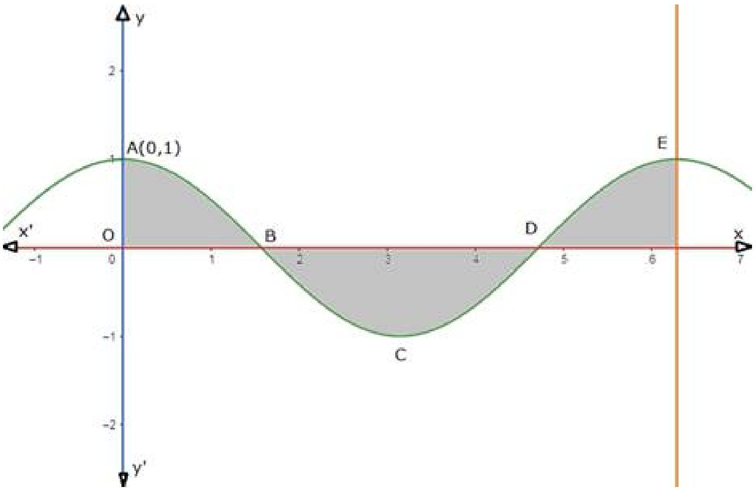
x = 0 ......(iii)

x = 2π .....(iv)

A table for values of y = cos x is: -

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	1.57

A rough sketch of the curves is given below: -



We have to find the area of shaded region.

Required area

= (shaded region ABOA + shaded region BCDB + shaded region DEFD)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= \int_0^{2\pi} y dx$ (As x is between (0,2 π) and the value of y varies)

$= \int_0^{\frac{\pi}{2}} (\cos x)\, dx + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\cos x)\, dx\right| + \int_{\frac{3\pi}{2}}^{2\pi} (\cos x)\, dx$ (as y = cos x)

On integrating we get,

$= [\sin x]_0^{\frac{\pi}{2}} + \left|[\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| + [\sin x]_{\frac{3\pi}{2}}^{2\pi}$

On applying the limits we get

$= \left(\sin\left(\frac{\pi}{2}\right) - \sin 0\right) + \left|\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right| + \left(\sin(2\pi) - \sin\left(\frac{3\pi}{2}\right)\right)$

= 1 - 0 + | - 1 - 1| + 0 - (- 1) = 4

Hence the area bounded by the curve y = cosx, x - axis and the ordinates x = 0 and x = 2π is equal to 4 square units.

24. Question

Show that the areas under the curves y = sin x and y = sin 2x between x = 0 and $x = \frac{\pi}{3}$ are in the ration 2:3.

Answer

Given equations are:

y = sin x .....(i)

y = sin 2x .....(ii)

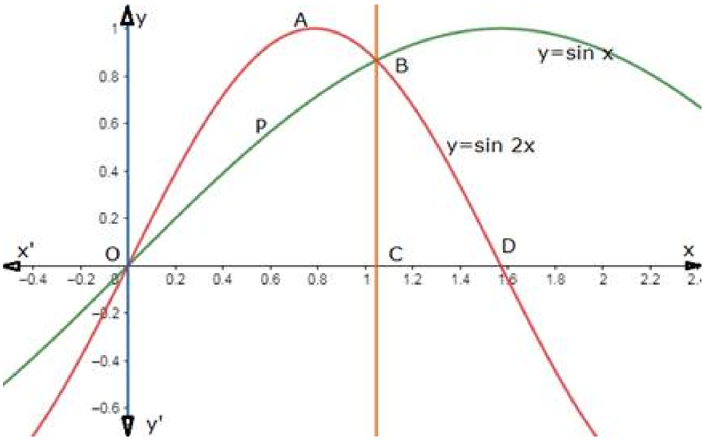
x = 0 ......(iii)

$x = \frac{\pi}{3}$ .....(iv)

A table for values of y = sin x and y = sin 2x is: -

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin x	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
sin 2x	0	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0

A rough sketch of the curves is given below: -



The area under the curve y = sin x , x = 0 and $x = \frac{\pi}{3}$ is

A_1 = (area of the region OPBCA)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$A_1 = \int_0^{\frac{\pi}{3}} \sin x\, dx$ (As x is between $\left(0, \frac{\pi}{3}\right)$ and the value of y varies)

$A_1 = \int_0^{\frac{\pi}{3}} (\sin x)\, dx$ (as y = sin x)

On integrating we get,

$A_1 = [-\cos x]_0^{\frac{\pi}{3}}$

On applying the limits we get

$A_1 = -\left(\cos\left(\frac{\pi}{3}\right) - \cos 0\right)$

$A_1 = -\left(\frac{1}{2} - 1\right) = \frac{1}{2}$

The area under the curve y = sin 2x , x = 0 and $x = \frac{\pi}{3}$ is

A_2 = (area of the region OABCO)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$A_2 = \int_0^{\frac{\pi}{3}} \sin 2x\, dx$ (As x is between $\left(0, \frac{\pi}{3}\right)$ and the value of y varies)

$A_2 = \int_0^{\frac{\pi}{3}} (\sin 2x)\, dx$ (as y = sin 2x)

On integrating we get,

$A_2 = \left[\frac{(-\cos 2x)}{2}\right]_0^{\frac{\pi}{3}}$

On applying the limits we get

$$A_2 = -\frac{1}{2}\left(\cos\left(\frac{2\pi}{3}\right) - \cos 0\right)$$

$$A_2 = -\frac{1}{2}\left(-\frac{1}{2} - 1\right) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$$

So the ratio of the areas under the curves y = sin x and y = sin 2x between x = 0 and $x = \frac{\pi}{3}$ are

$$A_1 : A_2 = \frac{1}{2} : \frac{3}{4} = 2 : 3$$

Hence showed

25. Question

Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between x = 0 and x = π.

Answer

Given equations are:

$y = \cos^2 x$ .....(i)

$y = \sin^2 x$ .....(ii)

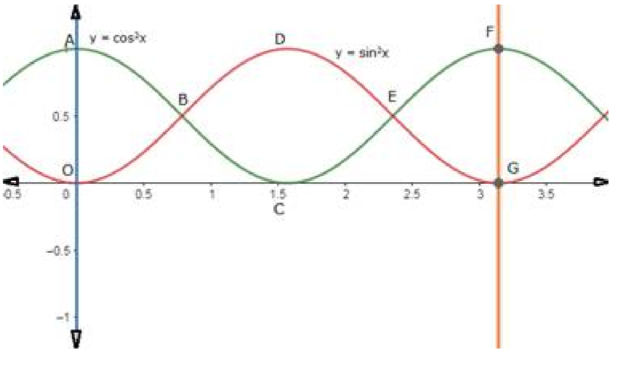
x = 0 ......(iii)

x = π .....(iv)

A table for values of $y = \cos^2 x$ and $y = \sin^2 x$ is: -

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos^2 x$	1	0.5	0.75	0.25	0
$\sin^2 x$	0	0.25	0.5	0.75	1

A rough sketch of the curves is given below: -



The area under the curve $y = \cos^2 x$, x = 0 and x = π is

A_1 = (area of the region OABCO + area of the region CEFGC)

A_1 = 2(area of the region CEFGC)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$A_1 = 2\int_{\frac{\pi}{2}}^{\pi} \cos^2 x \, dx$ (As x is between $\left(\frac{\pi}{3}, \pi\right)$ and the value of y varies)

Apply reduction formula:

$$\left[\int \cos^n x\,dx = \frac{n-1}{n}\int \cos^{n-2} x\,dx + \frac{(\cos^{n-1} x \sin x)}{n}\right]$$

On integrating we get,

$$A_1 = 2\left[\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(1)dx + \frac{\sin x \cos x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$A_1 = [(x) + \sin x \cos x]_{\frac{\pi}{2}}^{\pi}$$

On applying the limits we get

$$A_1 = (\pi + \sin\pi\cos\pi) - \left(\frac{\pi}{2} + \sin\frac{\pi}{2}\cos\frac{\pi}{2}\right)$$

$$A_1 = \frac{\pi}{2}$$

The area under the curve y = $\cos^2 x$, x = 0 and x = π is

A_2 = (area of the region OBDGEO)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$A_2 = \int_0^{\pi} \sin^2 x\,dx$ (As x is between $(0, \pi)$ and the value of y varies)

Apply reduction formula:

$$\left[\int \sin^n x\,dx = \frac{n-1}{n}\int \sin^{n-2} x\,dx + \frac{(\sin^{n-1} x \cos x)}{n}\right]$$

On integrating we get,

$$A_2 = \left[\frac{1}{2}\int_0^{\pi}(1)dx - \frac{\sin x \cos x}{2}\right]_0^{\pi}$$

$$A_2 = \frac{1}{2}\,[(x) - \sin x \cos x]_0^{\pi}$$

On applying the limits we get

$$A_2 = \frac{1}{2}(\pi - \sin\pi\cos\pi) - \frac{1}{2}(0 + \sin 0 \cos 0)$$

$$A_2 = \frac{\pi}{2}$$

Hence $A_1 = A_2$

Therefore the areas under the curves y = $\cos^2 x$ and y = $\sin^2$ x between x = 0 and x = π are equal.

26. Question

Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates x = ae and x = 0, where $b^2 = a^2 (1 - e^2)$ and e<1.

Answer

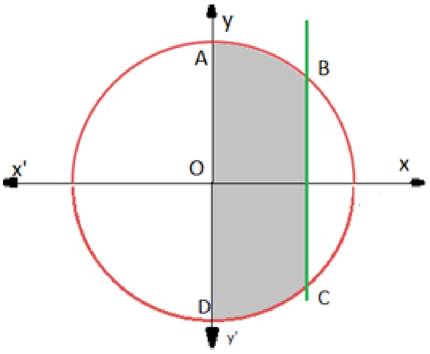
Given equations are:

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ ...... (1)

And x = ae, x = 0 ...... (2)

equation (1) represents an eclipse that is symmetrical about the x - axis and also about the y - axis, with center at origin and passes through (±a, 0) and (0, ±a).

A rough sketch is given as below: -



We have to find the area of shaded region.

Required area

= shaded region ABCDA

= 2 (shaded region OABO) (as it is symmetrical about the x - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 2\int_0^{ae} y\,dx$ (As x is between (0,ae) and the value of y varies)

$= 2\int_0^{ae} b\sqrt{1-\frac{x^2}{a^2}}\,dx$ (as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \Rightarrow y=b\sqrt{1-\frac{x^2}{a^2}}$)

$$=\frac{2b}{a}\int_0^{ae}\sqrt{a^2-x^2}\,dx$$

$$=\frac{2b}{a}\int_0^{ae}\sqrt{a^2-x^2}\,dx$$

Substitute $x = a\sin u \Rightarrow u = \sin^{-1}\left(\frac{x}{2}\right), dx = a\cos u\,du$

So the above equation becomes,

$$=\frac{2b}{a}\int_0^{ae}\sqrt{a^2-(a\sin u)^2}\,(a\cos u\,du)$$

$$=\frac{2b}{a}\int_0^{ae}a\cos u\sqrt{a^2-a^2\sin^2 u}\,du$$

We know, $a^2-a^2\sin^2 u = a^2(1-\sin^2 u) = a^2\cos^2 u$

So the above equation becomes,

$$=\frac{2b}{a}\int_0^{ae}a\cos u\sqrt{a^2\cos^2 u}\,du$$

$$=\frac{2b}{a}\int_0^{ae}a^2\cos^2 u\,du$$

Apply reduction formula:

$$\left[\int \cos^n u\, du = \frac{n-1}{n}\int \cos^{n-2} u\, du + \frac{(\cos^{n-1} u \sin u)}{n}\right]$$

On integrating we get,

$$= 2ab\left[\frac{1}{2}\int 1 du + \frac{(\cos u \sin u)}{2}\right]_0^{ae}$$

$$= 2ab\left[\frac{u}{2} + \frac{(\cos u \sin u)}{2}\right]_0^{ae} \quad \left[\because \int 1 du = u\right]$$

Undo the substituting, we get

$$= 2ab\left[\frac{\sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}{2}\right]_0^{ae}$$

$$= \frac{2ab}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{1-\frac{x^2}{a^2}} \times \frac{x}{a}\right]_0^{ae}$$

On applying the limits we get,

$$= ab\left[\left(\sin^{-1}\left(\frac{ae}{a}\right) + \frac{ae}{a}\sqrt{1-\frac{(ae)^2}{a^2}}\right) - \left(\sin^{-1}\left(\frac{0}{a}\right) + \frac{0}{a}\sqrt{1-\frac{0^2}{a^2}}\right)\right]$$

$$= ab\left[\left(\sin^{-1}(e) + e\sqrt{1-e^2}\right)\right]$$

Hence the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates x = ae and x = 0, where $b^2 = a^2(1 - e^2)$ and e<1 is equal to $ab\left[\left(\sin^{-1}(e) + e\sqrt{1-e^2}\right)\right]$ square units.

27. Question

Find the area of the minor segment of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{2}$.

Answer

Given equations are :

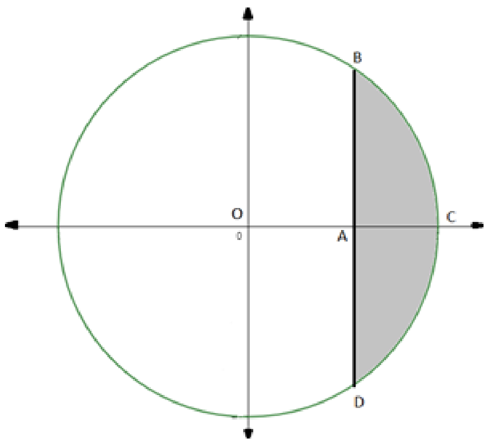
$x^2 + y^2 = a^2$ ...... (1)

$x = \frac{a}{2}$ ..... (2)

Equation (1) represents a circle with centre (0,0) and radius a, so it meets the axes at (±a,0), (0,±a).

Equation (2) represents a line parallel to y axis.

A rough sketch of the circle is given below: -



We have to find the area of shaded region.

Required area

= (shaded region BCDB)

= 2(shaded region ABCA) (as the circle is symmetrical about the x - axis as well as the y - axis)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 2\int_{\frac{a}{2}}^{a} y\,dx$ (As x is between $\left(\frac{a}{2}, a\right)$ and the value of y varies)

$= 2\int_{\frac{a}{2}}^{a} \sqrt{a^2 - x^2}\,dx$ (as $y = \sqrt{a^2 - x^2}$)

Substitute $x = a\sin u \Rightarrow u = \sin^{-1}\left(\frac{x}{a}\right), dx = a\cos u\,du$

So the above equation becomes,

$$= 2\int_{\frac{a}{2}}^{a} \sqrt{a^2 - (a\ \sin u)^2}\,(a\cos u\,du)$$

$$= 2\int_{\frac{a}{2}}^{a} a\ \cos u \sqrt{a^2 - a^2\sin^2 u}\,du$$

We know, $a^2 - a^2\sin^2 u = a^2(1 - \sin^2 u) = a^2\cos^2 u$

So the above equation becomes,

$$= 2\int_{\frac{a}{2}}^{a} a\cos u \sqrt{a^2\ \cos^2 u}\,du$$

$$= 2\int_{\frac{a}{2}}^{a} a^2\ \cos^2 u\,du$$

Apply reduction formula:

$$\left[\int \cos^n u\,du = \frac{n-1}{n}\int \cos^{n-2} u\,du + \frac{(\cos^{n-1} u \sin u)}{n}\right]$$

On integrating we get,

$$= 2a^2\left[\frac{1}{2}\int 1du + \frac{(\cos u \sin u)}{2}\right]_{\frac{a}{2}}^{a}$$

$$= 2a^2\left[\frac{u}{2} + \frac{(\cos u \sin u)}{2}\right]_0^a \left[\because \int 1du = u\right]$$

Undo the substituting, we get

$$= 2a^2\left[\frac{\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}{2}\right]_{\frac{a}{2}}^{a}$$

$$= 2a^2\left[\frac{\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{\sqrt{1-\frac{x^2}{a^2}}\times\frac{x}{a}}{2}\right]_{\frac{a}{2}}^{a}$$

$$= a^2\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]_{\frac{a}{2}}^{a}$$

On applying the limits we get,

$$= a^2\left[\left(\sin^{-1}\left(\frac{a}{a}\right)+\frac{a}{a}\sqrt{1-\frac{a^2}{a^2}}\right)-\left(\sin^{-1}\left(\frac{\frac{a}{2}}{a}\right)+\frac{\frac{a}{2}}{a}\sqrt{1-\frac{\left(\frac{a}{2}\right)^2}{a^2}}\right)\right]$$

$$= a^2\left[\left(\sin^{-1}(1)+1\sqrt{0}\right)-\left(\sin^{-1}\left(\frac{1}{2}\right)+\frac{1}{2}\sqrt{1-\frac{1}{4}}\right)\right]$$

$$= a^2\left(\frac{\pi}{2}-\left(\frac{\pi}{6}+\frac{1}{4}\sqrt{3}\right)\right)$$

$$= a^2\left(\frac{6\pi-2\pi}{12}-\frac{1}{4}\sqrt{3}\right)$$

$$= a^2\left(\frac{\pi}{3}-\frac{1}{4}\sqrt{3}\right)$$

$$= \frac{a^2}{12}\left(4\pi-3\sqrt{3}\right)$$

Hence the area of the minor segment of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{2}$ is equal to $\frac{a^2}{12}(4\pi-3\sqrt{3})$ square units.

28. Question

Find the area of the region bounded by the curve $x = at^2$, $y = 2at$ between the ordinates corresponding $t = 1$ and $t = 2$

Answer

Given equations are:

$x = at^2$ ...... (1)

$y = 2at$ ..... (2)

$t = 1$ ..... (3)

$t = 2$ ..... (4)

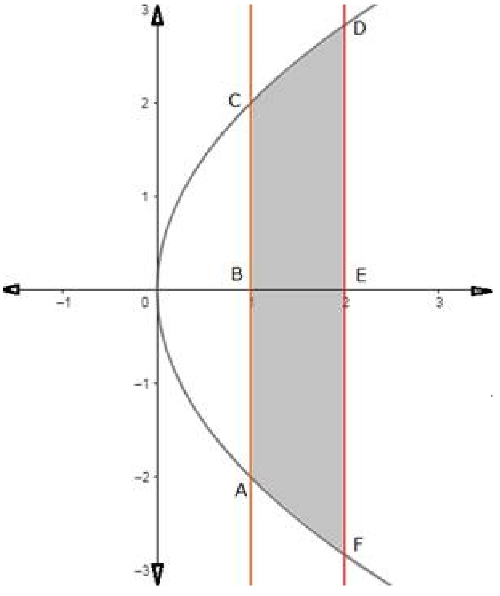
Equation (1) and (2) represents the parametric equation of the parabola.

Eliminating the parameter t, we get

$x = at^2 \Rightarrow x = a\left(\frac{y}{2a}\right)^2 \Rightarrow y^2 = 4ax$

This represents the Cartesian equation of the parabola opening towards the positive x - axis with focus at (a,0).

A rough sketch of the circle is given below: -



When t = 1, x = a

When t = 2, x = 4a

We have to find the area of shaded region.

Required area

= (shaded region ABCDEF)

= 2(shaded region BCDEB)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 2\int_a^{4a} y\,dx$ (As x is between $(a, 4a)$ and the value of y varies, here y is Cartesian equation of the parabola)

$= 2\int_a^{4a} \sqrt{4ax}\,dx$ (as $y^2 = 4ax$)

$= 4\sqrt{a}\int_a^{4a} \sqrt{x}\,dx$

On integrating we get,

$= 4\sqrt{a}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_a^{4a}$ (by applying power rule)

On applying the limits we get,

$= \frac{8\sqrt{a}}{3}\left[\left(\sqrt{(4a)^3}\right) - \left(\sqrt{(a)^3}\right)\right]$

$= \frac{8\sqrt{a}}{3}\left[8a\sqrt{a} - a\sqrt{a}\right]$

$= \frac{8\sqrt{a}(7a\sqrt{a})}{3} = \frac{56a^2}{3}$

Hence the area of the region bounded by the curve x = at^2, y = 2at between the ordinates corresponding t = 1 and t = 2 is equal to $\frac{56a^2}{3}$ square units.

29. Question

Find the area enclosed by the curve x = 3 cost, y = 2 sint

Answer

Given equations are x = 3 cost, y = 2 sint

These are the parametric equation of the eclipse.

Eliminating the parameter t, we get

$x = 3\ \text{cost} \Rightarrow \frac{x}{3} = \cos t \dots \dots (i)$

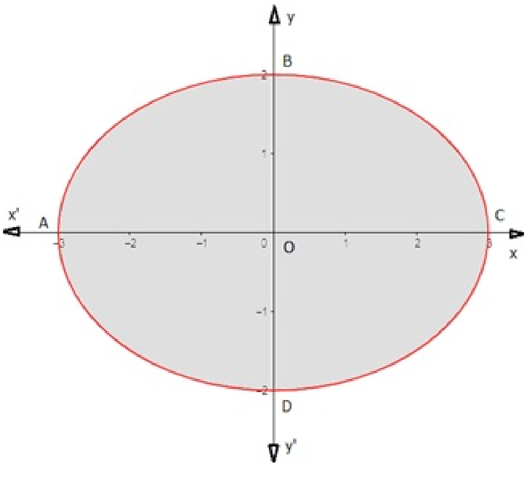
$y = 2\ \text{sint} \Rightarrow \frac{y}{2} = \sin t \dots \dots (ii)$

Squaring and adding equation (i) and (ii), we get

$\frac{x^2}{9} + \frac{y^2}{4} = \sin^2 t + \cos^2 t = 1$ (as $\sin^2 t + \cos^2 t = 1$)

This is Cartesian equation of the eclipse.

A rough sketch of the circle is given below: -



We have to find the area of shaded region.

Required area

= (shaded region ABCDA)

= 4(shaded region OBCO)

(the area can be found by taking a small slice in each region of width Δx, then the area of that sliced part will be yΔx as it is a rectangle and then integrating it to get the area of the whole region)

$= 4\int_0^3 y\,dx$ (As x is between $(0,3)$ and the value of y varies, here y is Cartesian equation of the eclipse)

$= 4\int_0^3 2\sqrt{1-\frac{x^2}{9}}\,dx$ (as $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{y^2}{4} = 1 - \frac{x^2}{9} \Rightarrow y = 2\sqrt{1-\frac{x^2}{9}}$)

$= 8\int_0^3 \frac{1}{3}\sqrt{9-x^2}\,dx$

$= \frac{8}{3}\int_0^3 \sqrt{9-x^2}\,dx$

Substitute $x = 3\sin u \Rightarrow u = \sin^{-1}\left(\frac{x}{3}\right), dx = 3\cos u\, du$

So the above equation becomes,

$$= \frac{8}{3}\int_0^3 \sqrt{9-(3\sin u)^2}\,(3\cos u\, du)$$

$$= \frac{8}{3}\int_0^3 3\cos u\sqrt{9-9\sin^2 u}\, du$$

We know, $9 - 9\sin^2 u = 9(1-\sin^2 u) = 9\cos^2 u$

So the above equation becomes,

$$= \frac{8}{3}\int_0^3 3\cos u\sqrt{9\cos^2 u}\, du$$

$$= \frac{8}{3}\int_0^3 9\cos^2 u\, du$$

Apply reduction formula:

$$\left[\int \cos^n u\, du = \frac{n-1}{n}\int \cos^{n-2} u\, du + \frac{(\cos^{n-1} u \sin u)}{n}\right]$$

On integrating we get,

$$= 24\left[\frac{1}{2}\int 1 du + \frac{(\cos u \sin u)}{2}\right]_0^3$$

$$= 24\left[\frac{u}{2} + \frac{(\cos u \sin u)}{2}\right]_0^3 \left[\because \int 1 du = u\right]$$

Undo the substituting, we get

$$= 24\left[\frac{\sin^{-1}\left(\frac{x}{3}\right)}{2} + \frac{\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)\sin\left(\sin^{-1}\left(\frac{x}{3}\right)\right)}{2}\right]_0^3$$

$$= 12\left[\sin^{-1}\left(\frac{x}{3}\right) + \sqrt{1-\frac{x^2}{9}} \times \frac{x}{3}\right]_0^3$$

On applying the limits we get,

$$= 12\left[\left(\sin^{-1}\left(\frac{3}{3}\right) + \frac{3}{3}\sqrt{1-\frac{3^2}{9}}\right) - \left(\sin^{-1}\left(\frac{0}{3}\right) + \frac{0}{3}\sqrt{1-\frac{0^2}{9}}\right)\right]$$

$$= 12\left[\left(\sin^{-1}(1) + \sqrt{0}\right) - 0\right] = 6\pi$$

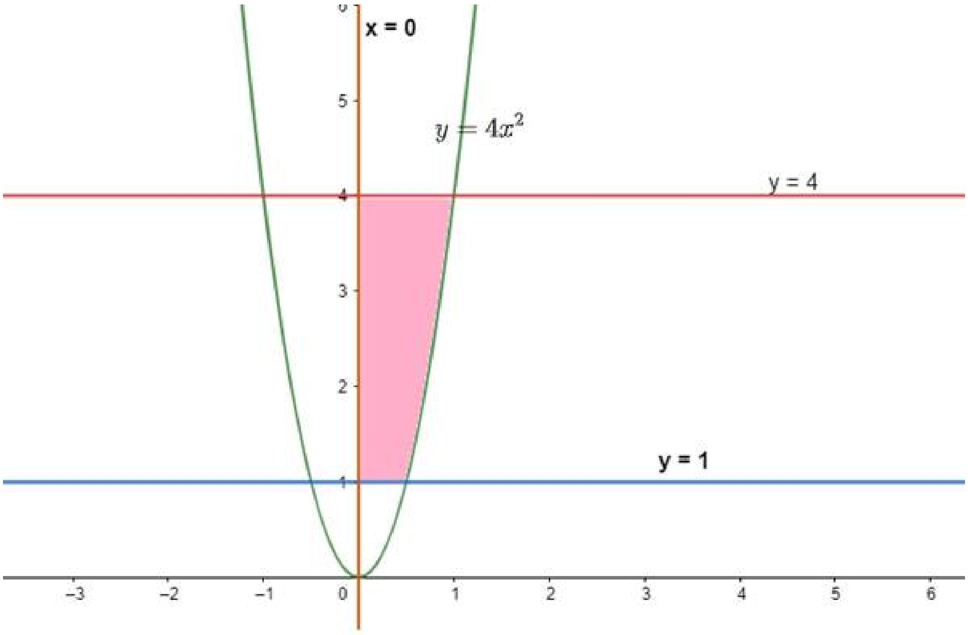
Hence the area enclosed by the curve x = 3 cost, y = 2 sintis equal to 6π square units.

Exercise 21.2

1. Question

Find the area of the region in the first quadrant bounded by the parabola $y = 4x^2$ and the lines x = 0, y = 1 and y = 4.

Answer



To find the area under two or more than two curves, the first crucial step is to find the INTERSECTION POINTS of the curves.

$\Longrightarrow y = 4x^2$ (Curve A), $y = 1$ (Line C)

$\Longrightarrow 1 = 4x^2$

$\Longrightarrow x = \frac{1}{2}$

The coordinates $(\frac{1}{2}, 1)$

$\Longrightarrow y = 4x^2$ (Curve A), $y = 1$ (Line B)

$\Longrightarrow y = 4x^2,\ y = 4$

$\Longrightarrow 4 = 4x^2$

$\Longrightarrow x = +\ 1$

Required Area can be calculated by breaking the problem into two parts.

I. Calculate Area under the curve A and Line C

II. Subtract the area enclosed by curve A and Line B from the above area.

Therefore, the areas are:

I. $\int_0^1 (4 - 4x^2).dx$ = Area enclosed by line C and curve A

$\Longrightarrow [4x]_0^1 - \left[\frac{4x^3}{3}\right]_0^1$

$\Longrightarrow \frac{8}{3}$

II. $\int_0^{\frac{1}{2}} (1 - 4x^2).dx$ = Area enclosed by curve A and Line B.

$\Longrightarrow [x]_0^{\frac{1}{2}} - \left[\frac{4x^3}{3}\right]_0^{\frac{1}{2}}$

$$\Rightarrow \frac{1}{2} - \frac{4\left[\frac{1}{2}\right]^3}{3}$$

$$\Rightarrow \frac{1}{3}$$

Now the required area under the curves:

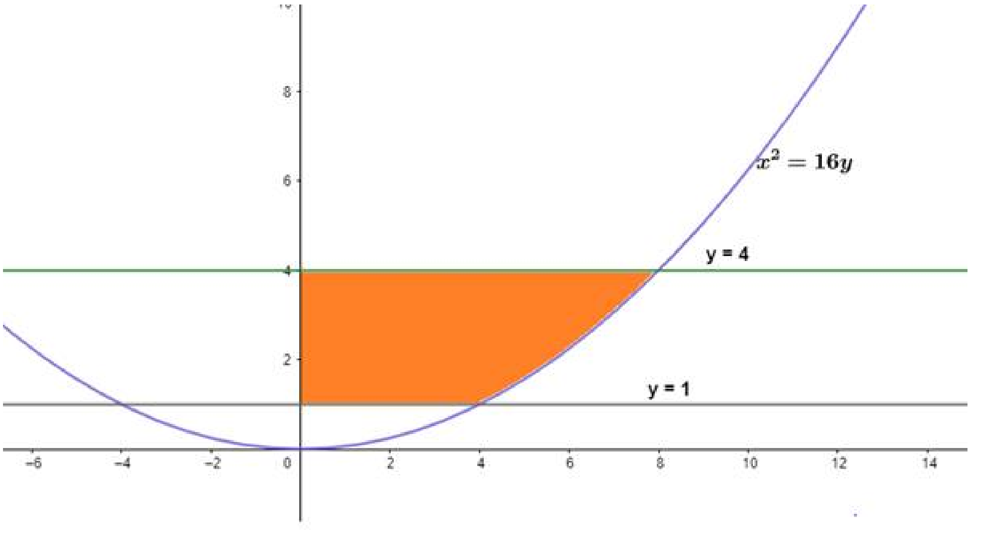
$$\Rightarrow \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Area bounded $= \frac{7}{3}$ square units.

2. Question

Find the area of the region bounded by $x^2 = 16y$, y = 1, y = 4 and the y – axis in the first quadrant.

Answer



To find the area under two or more than two curves, the first crucial step is to find the INTERSECTION POINTS of the curves.

$$y = \frac{x^2}{16} \text{ (Curve A)},$$

$$y = 4 \text{ (Line B)},$$

$$y = 1 \text{ (Line C)}$$

Between Curve A and Line C

$$\Rightarrow 16 = x^2$$

$$\Rightarrow x = 4$$

Between Curve A and Line B

$$\Rightarrow 4 = \frac{x^2}{16}$$

$$\Rightarrow x = \sqrt{64}$$

$$\Rightarrow x = 8$$

Required Area can be calculated by breaking the problem into two parts.

I. Calculate Area under the curve A and Line B.

II. Subtract the area enclosed by curve A and Line C from the above area.

I. $\int_0^8 \left(4-\frac{x^2}{16}\right).dx$ = Area under B and A

$$\Rightarrow [4x]_0^8 - \left[\frac{x^3}{16\times 3}\right]_0^8$$

$$\Rightarrow \frac{64}{3}$$

II. $\int_0^4 \left(1-\frac{x^2}{16}\right).dx$ = Area under C and A

$$\Rightarrow [x]_0^4 - \left[\frac{x^3}{16\times 3}\right]_0^4$$

$$\Rightarrow \frac{8}{3}$$

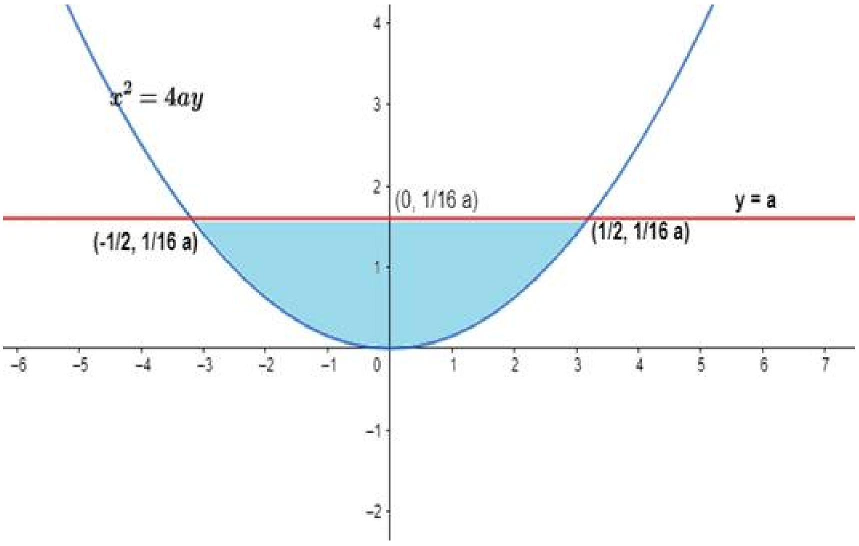
The required area under the curve

$$\Rightarrow \frac{64}{3}-\frac{8}{3} = \frac{56}{3} \text{ square units.}$$

3. Question

Find the area of the region bounded by x^2 = 4ay and its latus rectum.

Answer



This is a simple problem of the area under the two curves.

Step 1: Find the latus rectum and its intersection points with the parabola.

$$\Rightarrow x^2 = 4ay\ (1)$$

$$\Rightarrow y = \frac{x^2}{4a}\ (2)$$

Comparing it with the standard form of a parabola

Y = 4 A x^2

Where (0, A) is the coordinate of the focus of the parabola. And the latus rectum passes through this point and is perpendicular to the axis of symmetry.

Therefore, the equation of latus rectum is y = a.

Comparing equation (1) and (2)

$$\frac{1}{4a} = 4A$$

$$\Rightarrow A = \frac{1}{16a}$$

$\Rightarrow$ The equation of the latus rectum:

$$y = \frac{1}{16a}$$

Intersection points

$$\Rightarrow \frac{1}{16a} = \frac{x^2}{4a}$$

$$\Rightarrow x^2 = \frac{1}{4}$$

$$\Rightarrow x = \pm\frac{1}{2}$$

Step 2: Integrating the expression to find the area enclosed by the curves.

Since the latus rectum is above the parabola in the cartesian plane, the expression will be:

$$\int_{\frac{-1}{2}}^{\frac{1}{2}}\left[\frac{1}{16a} - \frac{x^2}{4a}\right]dx$$

$$\Rightarrow \left[\frac{x}{16a}\right]_{-1/2}^{1/2} - \left[\frac{x^3}{4a \times 3}\right]_{-1/2}^{1/2}$$

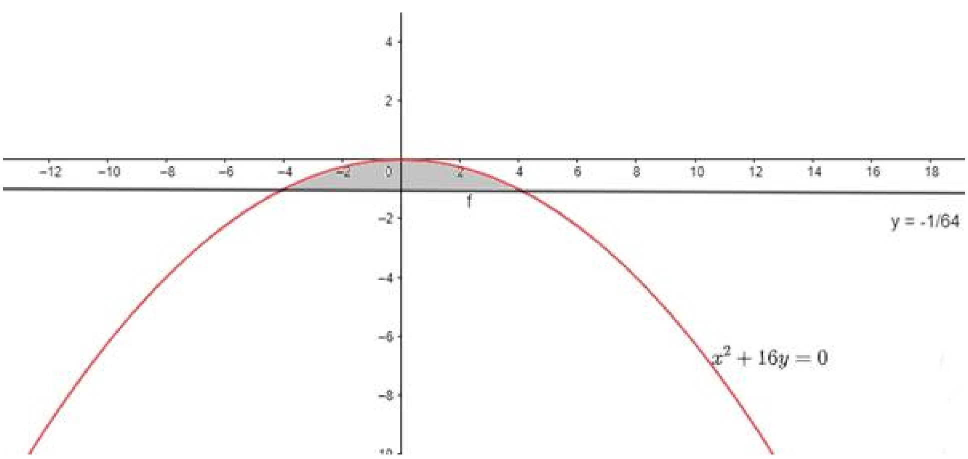
$$\Rightarrow \left[\frac{1}{16a}\right] - \left[\frac{1}{48a}\right]$$

$$\Rightarrow \frac{1}{24a}$$

4. Question

Find the area of the region bounded by $x^2 + 16y = 0$ and its latus rectum.

Answer



This is a simple problem of the area under the two curves.

Step 1: Find the latus rectum and its intersection points with the parabola.

As following the above questions procedure:

We find the equation of latus rectum as

$$y = -\frac{1}{64}$$

And the intersection points:

$$x = \pm\frac{1}{2}$$

Step 2: Integrating the expression to find the area enclosed by the curves.

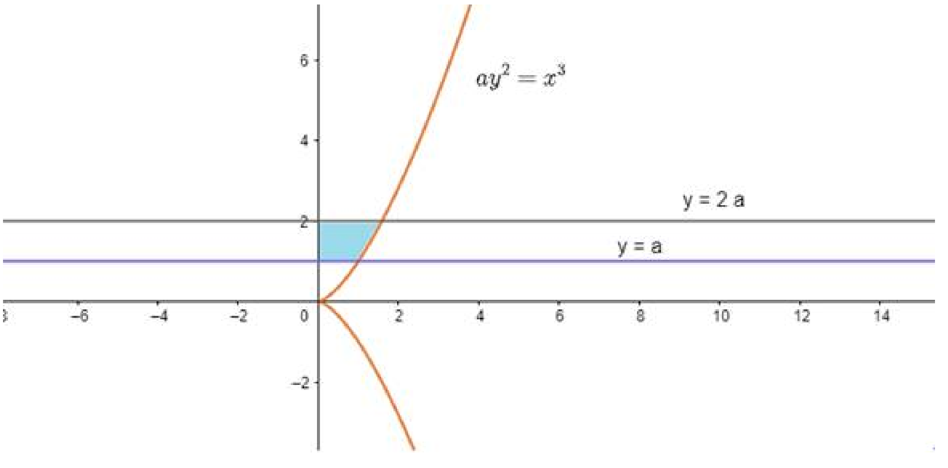
$$\int_{-1/2}^{1/2}\left[\frac{-x^2}{16} + \frac{1}{64}\right]dx$$

$$\Rightarrow \frac{1}{96}$$

5. Question

Find the area of the region bounded by the curve $ay^2 = x^3$, they y-axis and the lines y = a and y = 2a.

Answer



Similarly as problem 1,

Area of the bounded region = Area under (y = 2a) and ($ay^2 = x^3$) – Area under (y = a) and ($ay^2 = x^3$)

Area under (y = 2) and ($ay^2 = x^3$) =

$$\int_0^{1.58a}(2a - \sqrt{\frac{x^3}{a}})\,dx$$

$$\Rightarrow \left[2ax - \frac{2x^{\frac{5}{2}}}{5a^{\frac{1}{2}}}\right]_0^{1.58a}$$

$$\Rightarrow 2a(1.58a) - \frac{2(1.58a)^{\frac{5}{2}}}{5a^{\frac{1}{2}}}$$

Area under (y = a) and ($ay^2 = x^3$)

$$= \int_0^a(a - \sqrt{\frac{x^3}{a}})\,dx$$

$$\Rightarrow \left[ax - \frac{2x^{\frac{5}{2}}}{5a^{\frac{1}{2}}}\right]_0^a$$

$$\Rightarrow \left[a.a - \frac{2a^{\frac{5}{2}}}{5a^{\frac{1}{2}}}\right]$$

Required area:

$$2a(1.58a) - \frac{2(1.58a)^{\frac{5}{2}}}{5a^{\frac{1}{2}}} - \left[a^2 - \frac{2a^{\frac{5}{2}}}{5a^{\frac{1}{2}}}\right]$$

Simplify further and you will get the answer.

Exercise 21.3

1. Question

Calculate the area of the region bounded by the parabolas $y^2 = 6x$ and $x^2 = 6y$.

Answer

The given equations are,

$y^2 = 6x$

$y = \sqrt{6x} \ldots (i)$

And $x^2 = 6y$.

$y = \frac{x^2}{6} \ldots (ii)$

When y = 0 then x = 0,

Or $x = \sqrt{6y}$

Putting x value on $y^2 = 6x$,

$y^2 = 6\sqrt{6y}$

Or $\frac{y^2}{\sqrt{y}} = 6\sqrt{6}$

Or $y^{\frac{3}{2}} = 6^{\frac{3}{2}}$

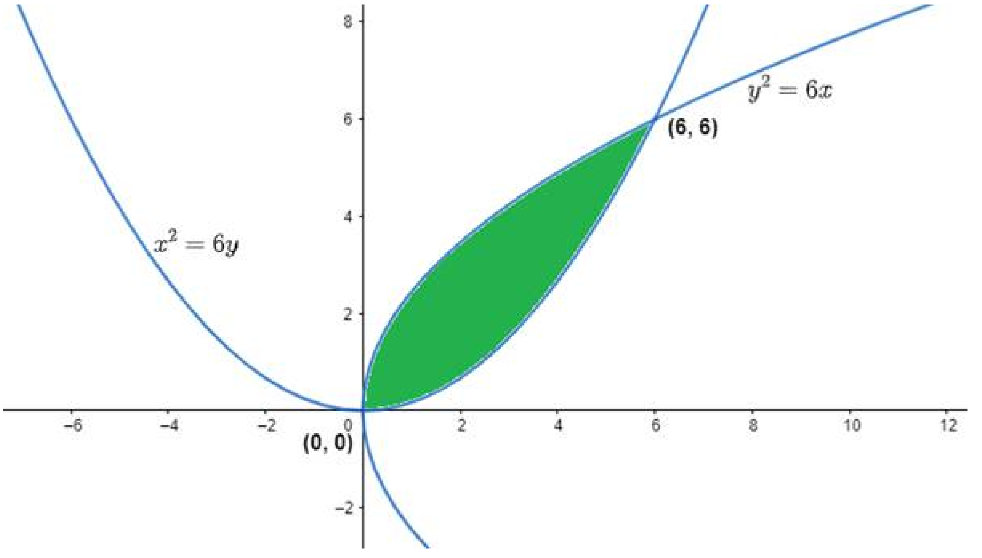
Or y = 6

When y = 0 then x = 0,

And When y = 6 then x = 6,

On solving these two equations, we get point of intersections.

The points are O (0,0) and A(6,6). These are shown in the graph below



Now the bounded area is the required area to be calculated, Hence,

Bounded Area, A = [Area between the curve (i) and x axis from 0 to 6] - [Area between the curve (ii) and x axis from 0 to 6]

$$A = \int_0^6 \sqrt{6x}dx - \int_0^6 \frac{x^2}{6}dx$$

$$A = \int_0^6 (\sqrt{6x} - \frac{x^2}{6})dx$$

On integrating the above definite integration,

$$A = \int_0^6 \left(\sqrt{6x} - \frac{x^2}{6}\right)dx$$

$$= \left[\sqrt{6}\frac{x^{3/2}}{3/2} - \frac{x^3}{18}\right]_0^6$$

$$= \left[\sqrt{6}\frac{(6)^{3/2}}{3/2} - \frac{(6)^3}{18}\right]$$

$$A = 12\text{sq. units}$$

Area of the region bounded by the parabolas $y^2 = 6x$ and $x^2 = 6y$ is 12sq. units.

2. Question

Find the area of the region common to the parabolas $4y^2 = 9x$ and $3x^2 = 16y$.

Answer

The given equations are,

$4y^2 = 9x$

$$y = \frac{3}{2}\sqrt{x} \dots (i)$$

And $3x^2 = 16y$.

$$y = \frac{3x^2}{16} \dots (ii)$$

Equating equation (i) and (ii)

$$\frac{3}{2}\sqrt{x} = \frac{3x^2}{16}$$

Or $x^{\frac{3}{2}} = 4^{\frac{3}{2}}$

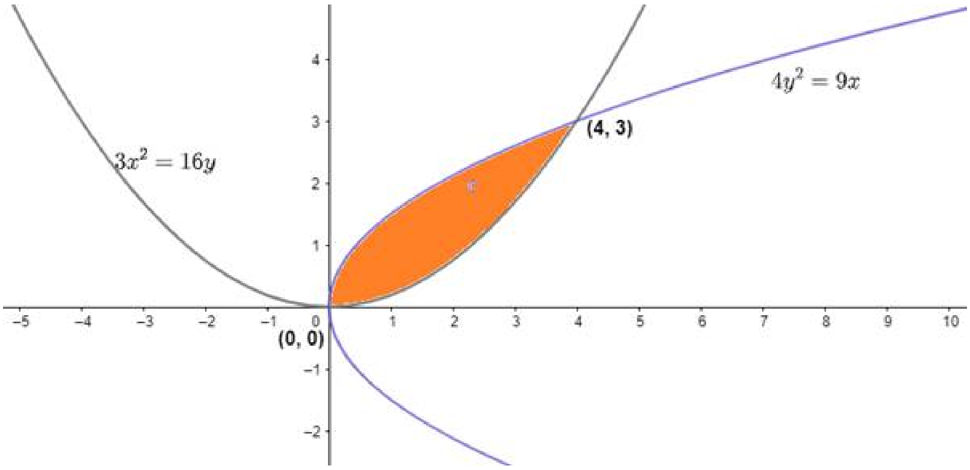
Or x = 4

When we put x = 4 in equation (i) then y = 3,

When we put x = 0 in equation (i) then y = 0,

On solving these two equations, we get the point of intersections.

The points are O (0, 0) and A(4,3). These are shown in the graph below



Now the bounded area is the required area to be calculated, Hence,

Bounded Area, A = [Area between the curve (i) and x axis from 0 to 4] - [Area between the curve (ii) and x axis from 0 to 4]

$$A = \int_0^4 \frac{3}{2}\sqrt{x}\,dx - \int_0^4 \frac{3x^2}{16}\,dx$$

$$A = \int_0^4 \left(\frac{3}{2}\sqrt{x} - \frac{3x^2}{16}\right)dx$$

On integrating the above definite integration,

The required area = $A = \int_0^4 \left[\frac{3\sqrt{x}}{2} - \frac{3x^2}{16}\right]dx$

$$= \left[x^{3/2} - \frac{x^3}{16}\right]_0^4$$

$$= \left[(4)^{3/2} - \frac{(4)^3}{16}\right]$$

$$= \left[8 - \frac{64}{16}\right]$$

$$= [8 - 4] = 4\text{ sq. units}$$

Area of the region common to the parabolas $4y^2 = 9x$ and $3\ x^2 = 16y$ is 4 sq. units

3. Question

Find the area of the region bounded by y = √x and y = x

Answer

The given equations are,

$y = \sqrt{x} \ldots (i)$

And y = x ...(ii)

Solving equation (i) and (ii)

$y^2 = x = y$

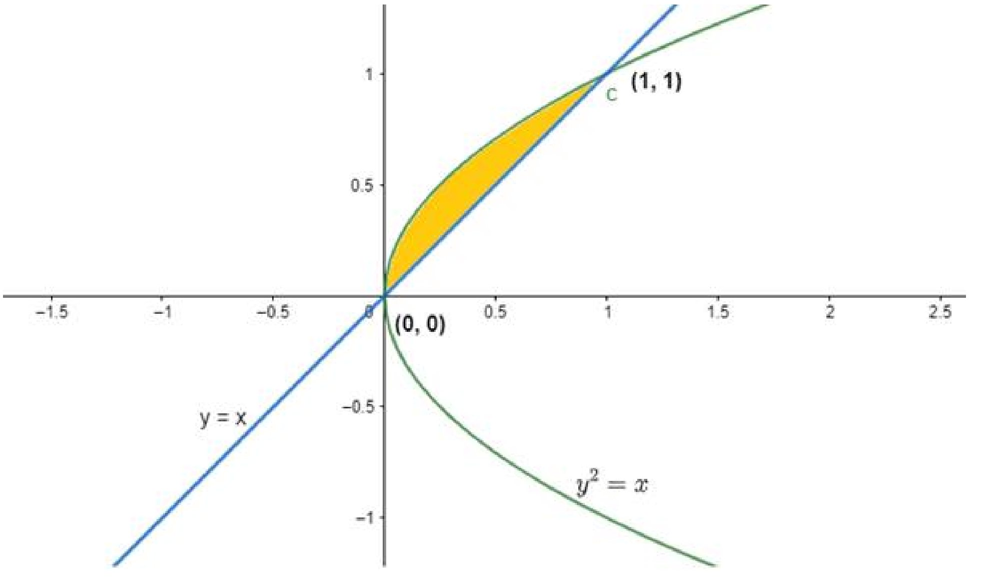
Or $y^2 = y$

Or y(y – 1) = 0

So, y = 0 or y = 1 and x = 0 or x = 1

On solving these two equations, we get the points of intersection.

The points are O (0, 0) and A(1,1). These are shown in the graph below



Now the bounded area is the required area to be calculated,

Hence, Bounded Area, A = [Area between the curve (i) and x axis from 0 to 1] - [Area between the curve (ii) and x axis from 0 to 1]

$$A = \int_0^1 \sqrt{x}\,dx - \int_0^1 x dx$$

$$A = \int_0^1 (\sqrt{x} - x)dx$$

On integrating the above definite integration,

$$= \int_0^1 (y_1 - y_2)dx$$

$$= \int_0^1 (\sqrt{x} - x)dx$$

$$= \left[\frac{2}{3}x\sqrt{x} - \frac{x^2}{2}\right]_0^1$$

$$= \left[\frac{2}{3}1\sqrt{1} - \frac{(1)^2}{2}\right] - [0]$$

$$= \left[\frac{2}{3} - \frac{1}{2}\right] = \frac{1}{6}\text{sq. units}$$

Area of the region bounded by $y = \sqrt{x}$ and Y = X is $\frac{1}{6}$ sq. units.

4. Question

Find the area bounded by the curve y = 4 – x^2 and the lined y = 0, y = 3.

Answer

The given equations are,

Y = 4 – x^2 ...(i)

Y = 0 ...(ii)

And y = 3 ...(iii)

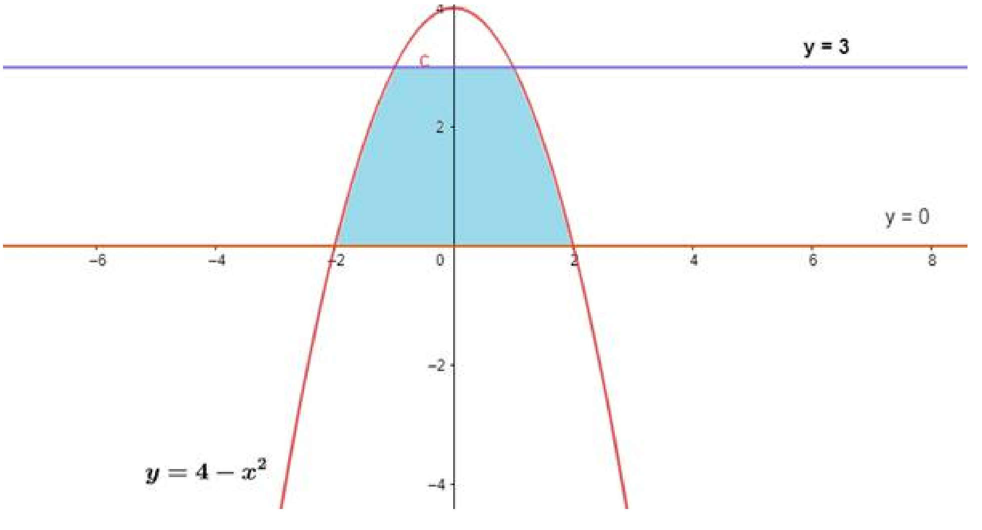
Equation (i) represents a parabola with vertex (0,4) and passes through (0,2),(0,02)

Equation (ii) is x - axis and cutting the parabola at C (2, 0)and D(– 2,0)

Equation (iii) is a line parallel to x - axis cutting the parabola at A(3,1)and B(– 3,1)

On solving these equations, we get point of intersections.

The points of intersections of a parabola with the other two lines are A(3,1), B(– 3,1), C(2,0) and D(– 2,0). These are shown in the graph below



Now the bounded area is the required area to be calculated,

Hence, Bounded Area, A = **2 times** [**Area between the equation (i) and y axis from y = 0 to y = 3**]

$$A = \int_0^3 \sqrt{4-y}\,dy$$

On integrating the above definite integration,

$$= -2\left[\frac{(4-y)^{3/2}}{3/2}\right]_0^3$$

$$= -2\frac{2}{3}\left[4^{3/2} - 1^{3/2}\right]$$

$$= \frac{28}{3}\text{ sq units}$$

The area bounded by the curve y = 4 – x^2 and the lined y = 0, y = 3 is $\frac{28}{3}$ sq units.

5. Question

Find the area of the region $\left\{(x,y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \le \frac{x}{a} + \frac{y}{b}\right\}$.

Answer

There are two equations involved in the question,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Represents an ellipse, symmetrical about both axis and cutting x - axis

at B (a,0) and (– a,0)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ Represents the area inside the ellipse

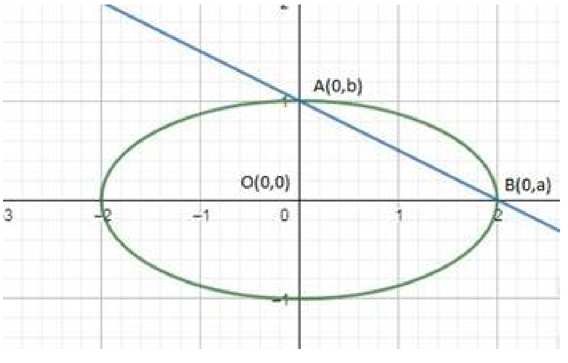
$y = \frac{b}{a}\sqrt{a^2 - x^2} \ldots (i)$

$\frac{x}{a} + \frac{y}{b} = 1$ Represents a straight line cutting x - axis at B(a,0)

$y = \frac{b}{a}(a - x)$ (ii)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} \ge 1$ Represents the area above the straight line.

Form the given these two equations; we get the point of intersections. The points are B(a,0) and A(0,b). These are shown in the graph below



The common area is the smaller area of an ellipse.

A = [Area between the curve (i) and x axis from 0 to a] – [Area between the curve (ii) and x axis from 0 to a]

$$A = \int_0^a \left[\frac{b}{a}\sqrt{a^2 - x^2} - \frac{b}{a}(a - x)\right] dx$$

$$= \frac{b}{a}\int_0^a 2\left[\sqrt{a^2 - x^2} - (a - x)\right] dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - ax + \frac{x^2}{2}\right]_0^a$$

$$= \frac{b}{a}\left[\left(\frac{a}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(1) - a^2 + \frac{a^2}{2}\right) - (0 + 0 + 0 + 0)\right]$$

$$= \frac{b}{a}\left[\frac{a^2}{2}.\frac{\pi}{2} - \frac{a^2}{2}\right]$$

$$= \frac{b}{a}\frac{a^2}{2}\left(\frac{\pi - 2}{2}\right)$$

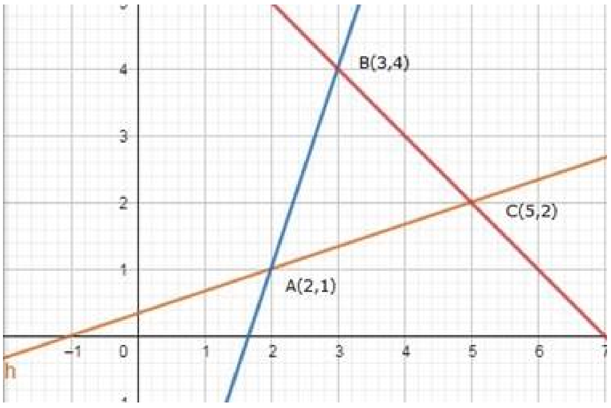
SO, the required area is $\frac{ab}{4}(\pi - 2)$ square units.

6. Question

Using integration find the area of the region bounded by the triangle whose vertices are (2,1), (3,4) and (5,2).

Answer

Here we have to find the area of the triangle whose triangle are A(2,1), B(3,4) and C(5,2) as shown below.



The equation of AB,

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 1 = \left(\frac{4 - 1}{3 - 1}\right)(x - 2)$$

$$y - 1 = \frac{3}{1}(x - 2)$$

Y = 3x – 5 ...(i)

The equation of BC,

$$y - 4 = \left(\frac{2 - 4}{5 - 3}\right)(x - 3)$$

$$= \frac{-2}{2}(x - 3)$$

$y = -x + 7$ ...(ii)

The equation of AC,

$$y - 1 = \left(\frac{2 - 1}{5 - 2}\right)(x - 2)$$

$$y - 1 = \frac{1}{3}(x - 2)$$

$$y = \frac{1}{3}x - \frac{2}{3} + 1$$

$$y = \frac{1}{3}x + \frac{1}{3} \text{ ...(iii)}$$

Now the required area (A) =

[(Area between line AB and x - axis) - (Area between line AC and x - axis) from x = 2 to x = 3]

+ [(Area between line BC and x - axis) - (Area between line AC and x - axis) from x = 3 to x = 5]

$$A = \int_2^3 (y_1 - y_3)dx + \int_3^5 (y_2 - y_3)dx$$

$$= \int_2^3 \left[(3x-5) - \left(\frac{1}{3}x + \frac{1}{3}\right)\right] dx + \int_3^5 \left[(-x+7) - \left(\frac{1}{3}x + \frac{1}{3}\right)\right] dx$$

$$= \int_2^3 \left[3x - 5 - \frac{1}{3}x + \frac{1}{3}\right] dx + \int_3^5 \left[-x + 7 - \frac{1}{3}x + \frac{1}{3}\right] dx$$

$$= \int_2^3 \left(\frac{8x}{3} - \frac{16}{3}\right) dx + \int_3^5 \left(-\frac{4}{3}x + \frac{20}{3}\right) dx$$

$$= \frac{8}{3}\left(\frac{x^2}{2} - 12x\right)_2^3 - \left(\frac{4x^2}{6} - \frac{20}{3}x\right)_3^5$$

$$= \frac{8}{3}\left[\left(\frac{9}{2} - 6\right) - (2-4)\right] - \left[\left(\frac{50}{3} - \frac{100}{3}\right) - (6-20)\right]$$

$$= \frac{8}{3}\left[-\frac{3}{2} + 2\right] - \left[-\frac{50}{3} + 14\right]$$

$$= \frac{4}{3} - \left[-\frac{8}{3}\right]$$

$= 4$ sq units.

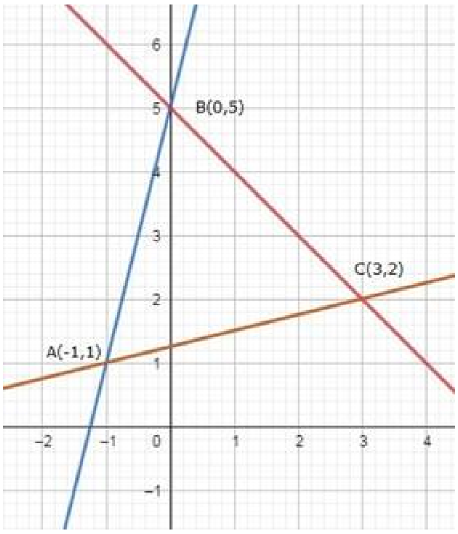
The area of the region bounded by the triangle whose vertices are (2,1), (3,4) and (5,2) is 4 sq. units

7. Question

Using integration, find the area of the region bounded by the triangle ABC whose vertices A, B, C are (– 1,1), (0,5) and (3,2) respectively.

Answer

We have to find the area of the triangle whose vertices are A(– 1,1), B (0,5), C(3,2) as shown below.



The equation of AB,

$$y = y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 1 = \left(\frac{5-1}{0+1}\right)(x+1)$$

$$y - 1 = \frac{4}{1}(x+1)$$

$$y = 4x + 4 + 1$$

Y = 4x + 5 ...(i)

The equation of BC,

$$y-5 = \left(\frac{2-5}{3-0}\right)(x-0)$$

$$= \frac{-3}{3}(x-0)$$

$$y-5 = -x$$

Y = 5 – x ...(ii)

The equation of AC,

$$y-1 = \left(\frac{2-1}{3+1}\right)(x+1)$$

$$y-1 = \frac{1}{4}(x+1)$$

$$y-1 = \frac{1}{4}x + \frac{1}{4} + 1$$

$$y = \frac{1}{4}(x+5)\ldots(iii)$$

Now the required area(A) =

[(Area between line AB and x - axis) - (Area between line AC and x - axis) from x = - 1 to x = 0]

+ [(Area between line BC and x - axis) - (Area between line AC and x - axis) from x = 0 to x = 3]

Say, $\text{Area A} = A_1 + A_2$

$$A_1 = \int_{-1}^{0}\left[(4x+5) - \frac{1}{4}(x+5)\right]dx$$

$$= \int_{-1}^{0}\left[4x + 5 - \frac{x}{4} - \frac{5}{4}\right]dx$$

$$= \int_{-1}^{0}\left(\frac{15}{4}x + \frac{15}{4}\right)dx$$

$$= \frac{15}{4}\left(\frac{x^2}{2} + x\right)_{-1}^{0}$$

$$= \frac{15}{4}\left[(0) - \left(\frac{1}{2} + 1\right)\right]$$

$$= \frac{15}{8}$$

And, $A_2 = \int_0^3 (y_2 - y_3)dx$

$$= \int_0^3\left[(5-x) - \left(\frac{1}{4}x + \frac{5}{4}\right)\right]dx$$

$$= \int_0^3\left[5 - x - \frac{1}{4}x - \frac{5}{4}\right]dx$$

$$= \int_0^3\left(-\frac{5}{4}x + \frac{15}{4}\right)dx$$

$= \frac{5}{4}\left(3x - \frac{x^2}{2}\right)_0^3$

$= \frac{5}{4}\left[9 - \frac{9}{2}\right] = \frac{45}{8}$

So the enclosed area of the triangle is $\frac{15}{8} + \frac{45}{8} = \frac{15}{2}$ sq Units.

8. Question

Using integration, find the area of the triangular region, the equations of whose sides are y = 2x + 1, y = 3x + 1 and x = 4.

Answer

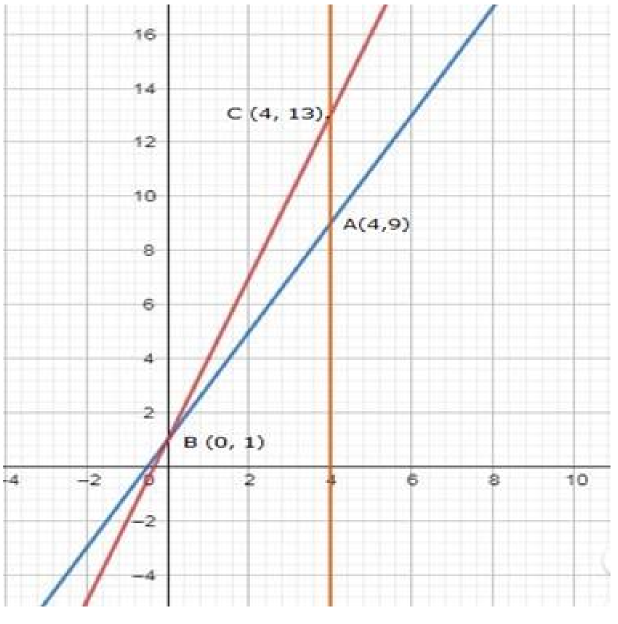
To find the area of the triangular region bounded by

y = 2x + 1 (Say, line AB) ...(i)

y = 3x + 1 (Say, line BC) ...(ii)

y = 4 (Say, line AC) ...(iii)

The sketch of the curves are drawn below,



Equation (i) represents a line passing through points B(0,1) and $\left(-\frac{1}{2}, 0\right)$.

Equation (ii) represents a line passing through (0,1) and $\left(-\frac{1}{3}, 0\right)$.

Equation (ii) represents a line parallel to y - axis passing through (4, 0).

Solving equation (i) and (ii) gives point B (0, 1).

Solving equation (ii) and (iii) gives point C (4, 13).

Solving equation (i) and (iii) gives point A (4, 9).

So, the Required area, A = (Region ABCA) = [Area between line BC and x - axis from x = 0 to x = 4]- [Area between line AB and x - axis from x = 0 to x = 4]

$A = \int_0^4 (y_1 - y_2)dx$

$= \int_0^4 [(3x + 1) - (2x + 1)]dx$

$= \int_0^4 x dx$

$$= \left[\frac{x^2}{2}\right]_0^4$$

= 8 sq.units

So the Required area is 8 sq. units.

9. Question

Find the area of the region $\{(x, y) : y^2 \le 8x,\ x^2 + y^2 \le 9\}$

Answer

To find area $\{(x, y): y^2 \le 8x,\ x^2 + y^2 \le 9\}$

$y^2 = 8x$ ...(i)

$x^2 + y^2 = 9$ ...(ii)

On solving the equation (i) and (ii),

Or, $x^2 + 8x = 9$

Or, $x^2 + 8x - 9 = 0$

Or, $(x + 9)(x - 1) = 0$

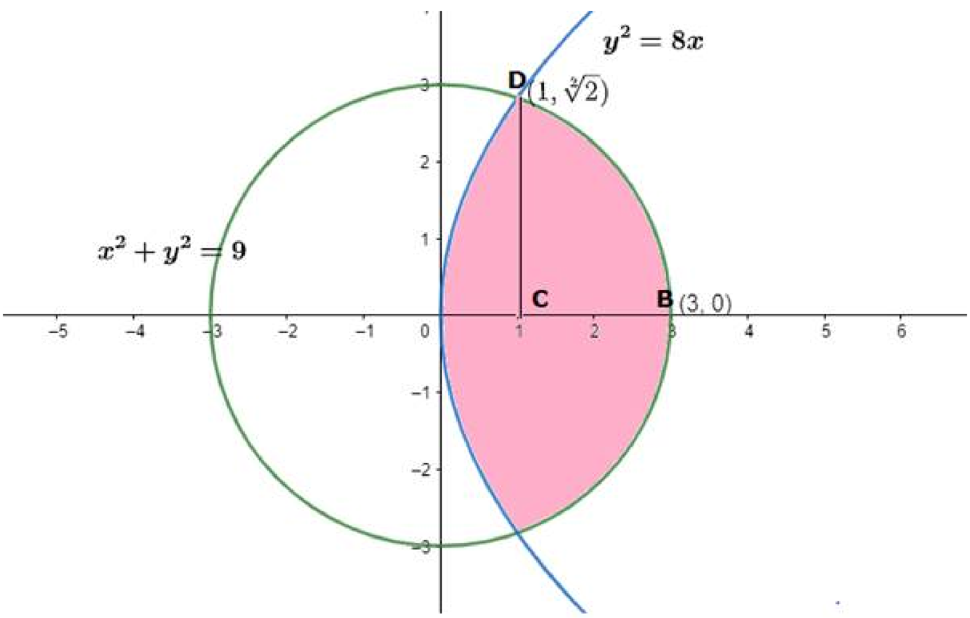
Or, $x = -9$ or $x = 1$

And when $x = 1$ then $y = \pm 2\sqrt{2}$

Equation (i) represents a parabola with vertex (0,0) and axis as x – axis, equation (ii) represents a circle with centre (0,0) and radius 3 units, so it meets area at (±3, 0), (0,±3).

Point of intersection of parabola and circle is $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$.

The sketch of the curves is as below:



Or, required area = 2(region ODCO + region DBCD)

$$= 2\left[\int_0^1 \sqrt{8x}\,dx + \int_1^3 \sqrt{9 - x^2}\,dx\right]$$

$$= 2\left[\left(2\sqrt{2.\frac{2}{3}x\sqrt{x}}\right)_0^1 + \left(\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right)_1^3\right]$$

$$= 2\left[\left(\frac{4\sqrt{2}}{3}.1.\sqrt{1}\right) + \left\{\left(\frac{3}{2}\sqrt{9-9} + \frac{9}{2}\sin^{-1}(1)\right) - \left(\frac{1}{2}\sqrt{9-1} + \frac{9}{2}\sin^{-1}\frac{1}{3}\right)\right\}\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \left\{\left(\frac{9}{2}.\frac{\pi}{2}\right) - \left(\frac{2\sqrt{2}}{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)\right\}\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \frac{9\pi}{4} - \sqrt{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

Hence, the required area is $2\left[\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$sq. units.

10. Question

Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

Answer

There are two equations,

$x^2 + y^2 = 16$ ...(i)

$y^2 = 6x$ ...(ii)

From (i) and (ii)

$X^2 + 6x = 16$

Or, $X^2 + 6x – 16 = 0$

Or, $(x + 8)(x – 2) = 0$

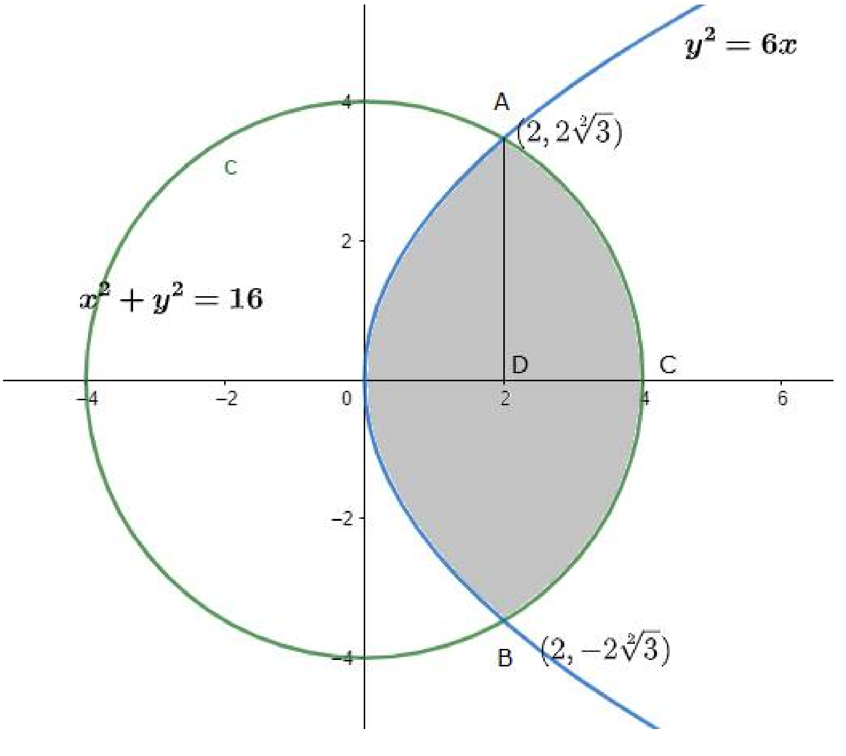
Or, x = – 8 or x = 2

And when x = 2 then $y = \pm 2\sqrt{3}$

Equation (i) represents a circle with centre (0,0) and radius 4 units, so it meets x - axis at (±4,0) and equation (ii) represents a parabola with vertex (0,) and axis as x - axis

Points of intersection of parabola and circle are $(2,2\sqrt{3})$ and $(2, – 2\sqrt{3})$.

The sketch of the two curves are drawn below,



The shaded region represents the required area.

Required area = Region OBCAO

Required area = 2 (region ODAO + region DCAD)

$$A = 2\left(\int_0^2 y_1 dx + \int_2^4 y_2 dx\right)$$

$$= 2\left[\int_0^2 \sqrt{6x}dx + \int_2^4 \sqrt{16 - x^2}dx\right]$$

$$= 2\left[\left\{\sqrt{6}.\frac{2}{3}x\sqrt{x}\right\}_0^2 + \left\{\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right\}_2^4\right]$$

$$= 2\left[\frac{4}{3}\sqrt{12} + \left\{(0 + 8\sin^{-1}(1)) - \left(1.\sqrt{12} + 8\sin^{-1}\left(\frac{1}{2}\right)\right)\right\}\right]$$

$$= 2\left[\frac{8\sqrt{3}}{3} + \left\{\left(8.\frac{\pi}{2}\right) - \left(2\sqrt{3} + \frac{\pi}{6}\right)\right\}\right]$$

$$= 2\left\{\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right\}$$

$$= 2\left\{\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right\}$$

$$= 2\left\{\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right\}$$

So, the required area is $2\left\{\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right\}$ sq units.

11. Question

Find the area of the region between circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

Answer

The given equations are,

$x^2 + y^2 = 4$ ...(i)

$(x - 2)^2 + y^2 = 4$ ...(ii)

Equation (i) is a circle with centre O at origin and radius 2.

Equation (ii) is a circle with centre C (2,0) and radius 2.

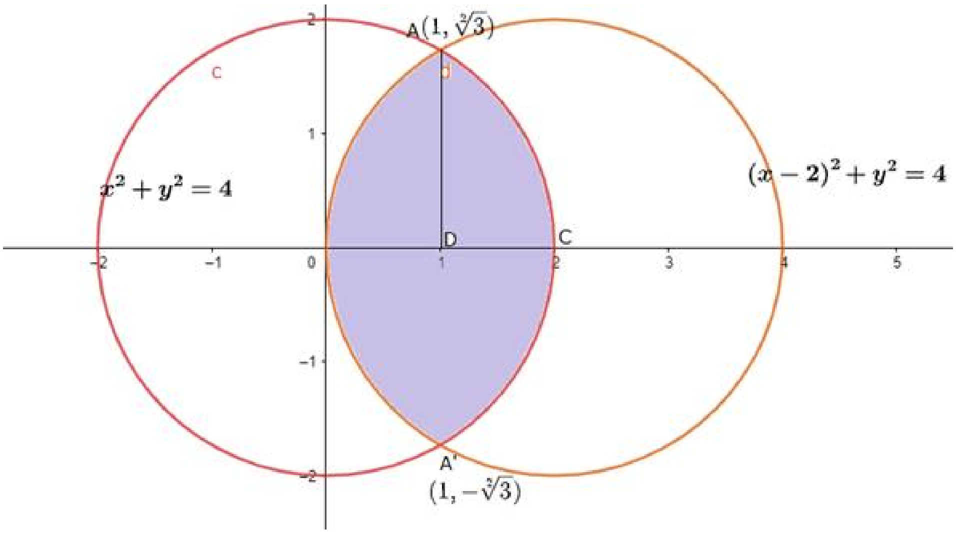
On solving these two equations, we have

$(x - 2)^2 + y^2 = x^2 + y^2$

Or $x^2 - 4x + 4 + y^2 = x^2 + y^2$

Or x = 1 which gives $y \pm \sqrt{3}$

Thus, the points of intersection of the given circles are A $(1, \sqrt{3})$ and A’ $(1, -\sqrt{3})$ as show in the graph below



Now the bounded area is the required area to be calculated, Hence,

Required area of the enclosed region OACA’O between circle

A = [area of the region ODCAO]

= 2 [area of the region ODAO + area of the region DCAD]

$= 2\left(\int_0^1 y dx + \int_1^2 y dx\right)$

$= 2\left[\int_0^1 \sqrt{4-(X-2)^2} dx + \int_1^2 \sqrt{4-x^2} dx\right]$ {from (i) and (ii)}

$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(X-2)^2} + \sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}x\sin^{-1}\frac{x}{2}\right]_1^2$

$= \left[(x-2)\sqrt{4-(X-2)^2} + \sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + \left[x\sqrt{4-x^2} + \frac{1}{2}x\sin^{-1}\frac{x}{2}\right]_1^2$

$= \left[\left(-\sqrt[2]{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(1)\right] + \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$

$= \left[\left(-\sqrt{3} - 4x\frac{\pi}{6}\right) + 4x\frac{\pi}{2}\right] + \left[4x\frac{\pi}{2} - \sqrt{3} - 4x\frac{\pi}{6}\right]$

$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right)$

$= \frac{8\pi}{3} - 2\sqrt{3}$squareunits

The area of the region between circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$ is $\frac{8\pi}{3} - 2\sqrt{3}$squareunits

12. Question

Find the area of the region included between the parabola $y^2 = x$ and the line x + y = 2.

Answer

To find region enclosed by

$y^2 = x$ ...(i)

And x + y = 2 ...(ii)

From equation (i) and (ii),

$y^2 + y - 2 = 0$

Or, $(y + 2)(y - 1) = 0$

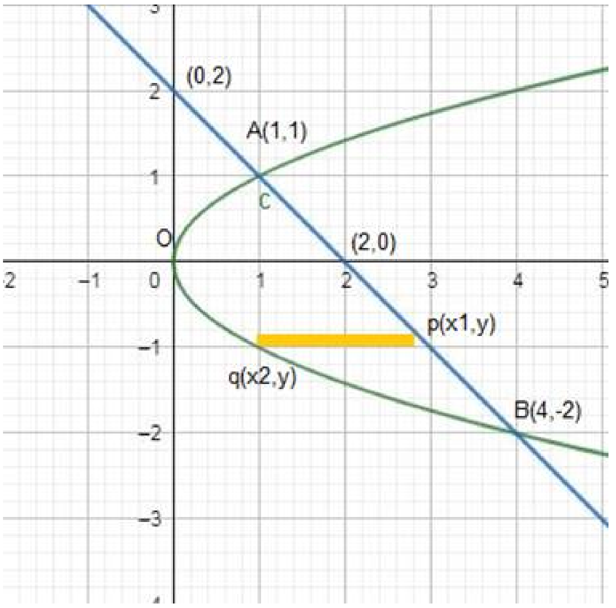
Or, $y = -2, 1$

$\therefore x = 4, 1$

Equation $y^2 = x$ represents a parabola with vertex at origin and its axis as x - axis

Equation $x + y = 2$ represents a line passing through (2,0) and (0,2)

On solving these two equations, we get point of intersections. The points of intersection of line and parabola are (1,1) and (4, – 2) These are shown in the graph below



Shaded region represents the required area. We slice it in rectangles of width Δy and length = $(x_1 - x_2)$.

Area of rectangle = $(x_1 - x_2)\Delta y$.

Required area of Region AOBA

$$= \int_{-2}^{1} (x_1 - x_2)dy$$

$$= \int_{-2}^{1} (2 - y - y^2)dy$$

$$= \left[2y - \frac{y_2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$= \left[\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-1 - 2 + \frac{8}{3}\right)\right]$$

$$= \left[\left(\frac{12 - 3 - 2}{6}\right) - \left(\frac{-12 - 6 + 8}{3}\right)\right]$$

$$= \frac{7}{6} + \frac{10}{3}$$

$$= \frac{9}{2} \text{ sq. units}$$

The area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

is $\frac{9}{2}$ sq. units

13. Question

Draw a rough sketch of the region $\{(x, y) : y^2 \le 3x, 3x^2 + 3y^2 \le 16\}$ and find the area enclosed by the region using method of integration.

Answer

To find area of region $\{(x, y) : y^2 \le 3x, 3x^2 + 3y^2 \le 16\}$

The given equations are,

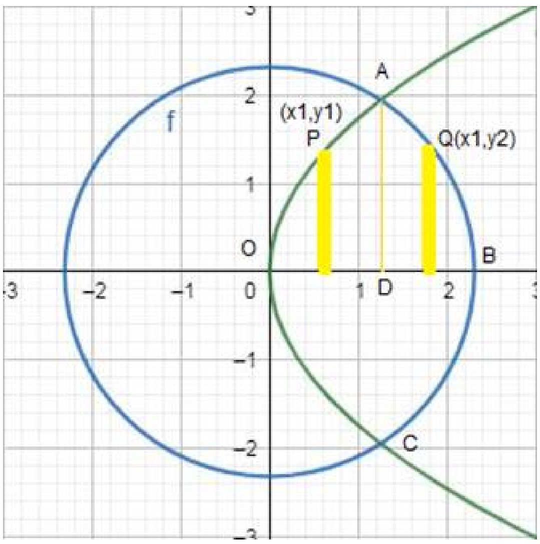
$y^2 = 3x$ ...(i)

And $3x^2 + 3y^2 = 16$

$x^2 + y^2 = \frac{16}{3}$ ...(ii)

Equation (i) represents a parabola with vertex (0,0) and axis as x - axis,

Equation (ii) represents a circle with centre (0,0) and radius $4/\sqrt{3}$ and meet at A $\left(\pm\frac{4}{\sqrt{3}}\right)$ and B $\left(0, \pm\frac{4}{\sqrt{3}}\right)$ A rough sketch of the region $\{(x, y) : y^2 \le 3x, 3x^2 + 3y^2 \le 16\}$ is drawn below



Required area = Region OCBAO

= 2(area of Region)

= 2(area of Region OBAO)

= 2(area of Region ODAO + area of Region DBAD)

$$= 2\left[\int_0^a \sqrt{3x}\,dx + \int_a^{\frac{4}{\sqrt{3}}} \sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - x^2}\,dx\right]$$

$$= 2\left[\left(\sqrt{3}.\frac{2}{3}x\sqrt{x}\right)_0^a + \left(\frac{x}{2}\sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - x^2} + \frac{16}{6}\sin^{-1}\frac{x\sqrt{3}}{4}\right)_a^{\frac{4}{\sqrt{3}}}\right]$$

$$= 2\left[\left(\frac{2}{\sqrt{3}}a\sqrt{a}\right) + \left\{\left(0 + \frac{8}{3}\sin^{-1}(1)\right) - \left(\frac{a}{2}\sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - a^2} + \frac{8}{3}\sin^{-1}\frac{a\sqrt{3}}{4}\right)\right\}\right]$$

$$= \frac{4}{\sqrt{3}}a^{\frac{3}{2}} + \frac{8\pi}{3} - a\sqrt{\frac{16}{3} - a^2} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)$$

$$= \frac{-9 + \sqrt{273}}{6}\,\text{sq.units}$$

The area enclosed by the region is $\frac{-9+\sqrt{273}}{6}$ sq. units

14. Question

Draw a rough sketch of the region {(x,y) : $y^2 \le 5x$, $5x^2 + 5y^2 \le 36$} and find the area enclosed by the region using the method of integration.

Answer

To find the area enclosed by the region{(x,y) : $y^2 \le 5x$, $5x^2 + 5y^2 \le 36$}

The given equations are,

$y^2 = 5x$ ...(i)

And $5x^2 + 5y^2 = 36$ ...(ii)

$x^2 + y^2 = \frac{36}{5}$

Substituting the value of y^2 from (i) into (ii)

$5x^2 + 25x = 36$

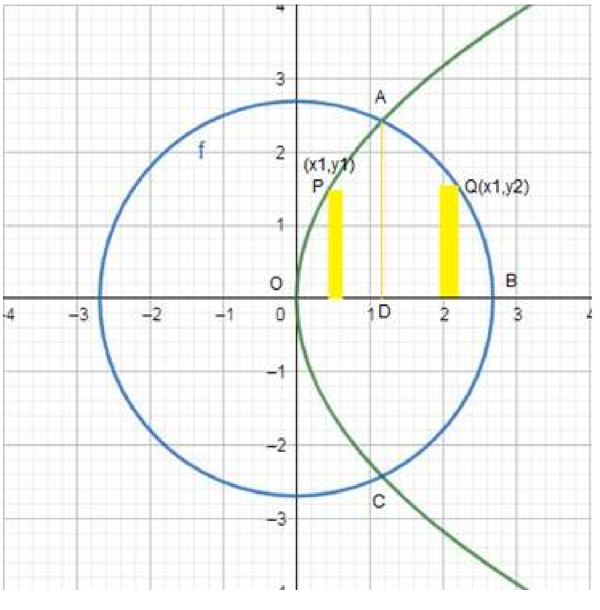
$5x^2 + 25x - 36 = 0$

$x = \frac{-25 \pm \sqrt{1345}}{10}$

Equation (i) represents a parabola with vertex (0, 0) and axis as x - axis.

Equation (ii) represents a circle with centre (0, 0) and radius 6/√5 and meets axes at $\left(\pm\frac{6}{\sqrt{5}}, 0\right)$ and $\left(0, \pm\frac{6}{\sqrt{5}}\right)$. X ordinate of the point of intersection of circle and parabola is A where a = $\frac{-25+\sqrt{1345}}{10}$.

A rough sketch of curves is: -



Required area = Region OCBAO

= 2 (Region OBAO)

= 2 (Region ODAO + Region DBAD)

$$= 2\left[\int_0^a \sqrt{5x}dx + \int_a^{\frac{6}{\sqrt{5}}} \sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - x^2}dx\right]$$

$$= 2\left[\left(\sqrt{5}.\frac{2}{3}x\sqrt{x}\right)_0^a + \left(\frac{x}{2}\sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - x^2}\frac{36}{10}\sin^{-1}\left(\frac{x\sqrt{5}}{6}\right)_a^{\frac{6}{\sqrt{5}}}\right)\right]$$

$$= \frac{4\sqrt{5}}{3}a\sqrt{a} + 2\left\{\left(0 + \frac{18}{5}.\frac{\pi}{2}\right) - \left(\frac{a}{2}\sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - a^2} + \frac{18}{5}\sin^{-1}\left(\frac{a\sqrt{5}}{6}\right)\right)\right\}$$

$$= \frac{4\sqrt{5}}{3}a^{\frac{3}{2}} + \frac{18\pi}{5} - a\sqrt{\frac{36}{5} - a^2} - \frac{36}{5}\sin^{-1}\left(\frac{a\sqrt{5}}{6}\right)$$

$$= \frac{125 + \sqrt{1345}}{10}$$

The area enclosed by the region $\{(x,y): y^2 \le 5x, 5x^2 + 5y^2 \le 36\}$ is $\frac{125+\sqrt{1345}}{10}$ sq. Units

15. Question

Draw a rough sketch and find the area of the region bounded by the two parabolas $y^2 = 4x$ and $x^2 = 4y$ by using methods of integration.

Answer

To find the area bounded by

$y^2 = 4x$

$y = 2\sqrt{x}$ ...(i)

And $x^2 = 4y$

$y = \frac{x^2}{4}$ ...(ii)

On solving the equation (i) and (ii),

$(\frac{x^2}{4})^2 = 4x$

Or, $x^4 - 64x = 0$

Or, $x(x^3 - 64) = 0$

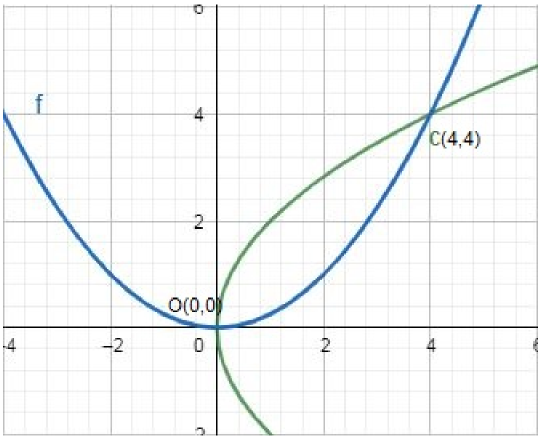
Or, x = 0, 4

Then y = 0,4

Equation (i) represents a parabola with vertex (0,0) and axis as x – axis. Equation (ii) represents a parabola with vertex (0,0) and axis as y - axis.

Points of intersection of the parabola are (0,0) and (4,4).

A rough sketch is given as: -



Now the bounded area is the required area to be calculated, Hence,

Bounded Area, A = [Area between the curve (i) and x axis from 0 to 4] – [Area between the curve (ii) and x axis from 0 to 4]

$$A = \int_0^4 2\sqrt{x}dx - \int_0^4 \frac{x^2}{4}dx$$

$$A = \int_0^6 (2\sqrt{x} - \frac{x^2}{4})dx$$

On integrating the above definite integration,

$$A = \int_0^4 2\sqrt{x} - \frac{x^2}{4}dx$$

$$= \left[2\frac{x^{3/2}}{3/2} - \frac{x^3}{12}\right]_0^4$$

$$= \left[2\frac{(4)^{3/2}}{3/2} - \frac{(4)^3}{18}\right]$$

$$= \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3}\text{sq. units}$$

Area of the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$ is $\frac{16}{3}$sq. units.

16. Question

Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Answer

To find area enclosed by

$Y^2 = 4ax$

$y = 2\sqrt{ax}$ ...(i)

And $X^2 = 4by$

$y = \frac{x^2}{4b}$ ...(ii)

On solving the equation (i) and (ii),

$$\frac{x^4}{16b^2} = 4ax$$

Or, $x^4 - 64ab^2x = 0$

Or, $x(x^3 - 64ab^2) = 0$

Or, x = 0 and x = $4\sqrt[3]{ab^2}$

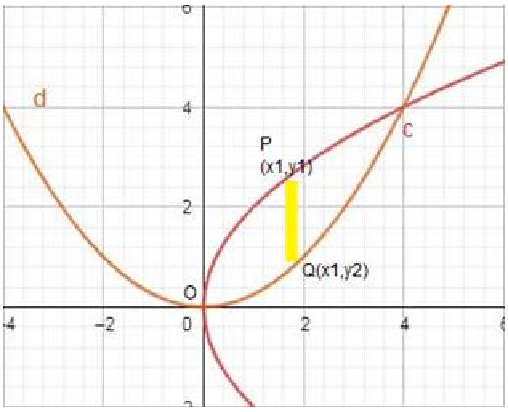
Then y = 0 and y = $4\sqrt[3]{a^2b}$

Equation (i) represents a parabola with vertex (0,0) and axis as x–axis,

Equation (ii) represents a parabola with vertex (0,0) and axis as x - axis,

Points of intersection of parabolas are O (0,0) and $C\left(4\sqrt[3]{ab^2}, 4\sqrt[3]{a^2b}\right)$

These are shown in the graph below: -



The shaded region is required area, and it is sliced into rectangles of width and length $(y^1 - y^2)\Delta X$.

This approximation rectangle slides from $x = 0$ x = 0 to $x = 4\sqrt[3]{a^2b}$, so

Required area = Region OQCPO

$$= \int_0^{4\sqrt[3]{a^2b}} (y_1 - y_2)dx$$

$$= \int_0^{4\sqrt[3]{a^2b}} \left(2\sqrt{a}.\sqrt{x} - \frac{x^2}{4b}\right) dx$$

$$= \left[2\sqrt{a}.\frac{2}{3}x\sqrt{x} - \frac{x^3}{12b}\right]_0^{4\sqrt[3]{a^2b}}$$

$$= \frac{32\sqrt{a}}{3}a\frac{1}{3}b\frac{2}{3}a\frac{1}{6}b\frac{1}{3} - \frac{64ab^2}{12b}$$

$$= \frac{32}{3}ab - \frac{16}{3}ab$$

$$= \frac{16}{3}ab \text{ sq. units}$$

The area included between the parabolas$y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16}{3}ab$ sq. units

17. Question

Prove that the area in the first quadrant enclosed by the axis, the line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$ is π/3.

Answer

To find an area in the first quadrant enclosed by the x – axis,

$x = \sqrt{3}y$

$x^2 + y^2 = 4$

Or $(\sqrt{3}y)^2 + y^2 = 4$

Or $4y^2 = 4$

Or $y = \pm 1$

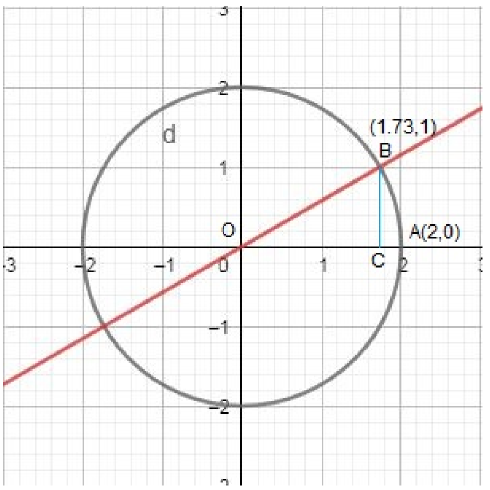
And $x = \pm\sqrt{3}$

Equation (i) represents a line passing through (0,0), (– √3, – 1), (√3,1).

Equation (ii) represents a circle centre (0,0) and passing through (±2,0), (0,±2).

Points of intersection of line and circle are (– √3, – 1) and (√3,1).

These are shown in the graph below: -



Required enclosed area = Region OABO

= Region OCBO + Region ABCA

$$= \int_0^{\sqrt{3}} y_1 dx + \int_{\sqrt{3}}^{2} y_2 dx$$

$$= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}} dx + \int_{\sqrt{3}}^{2} \sqrt{4 - x^2} dx$$

$$= \left(\frac{x^2}{2\sqrt{3}}\right)_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \left(\frac{3}{2\sqrt{3}} - 0\right) + \left[(0 + 2\sin^{-1}(1)) - \left(\frac{\sqrt{3}}{2}.1 + 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$$

$$= \frac{\sqrt{3}}{2} + 2.\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2.\frac{\pi}{3}$$

$$= \frac{\pi}{3} \text{sq. units}$$

Hence proved that the area in the first quadrant enclosed by the axis, the line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$ is π/3.

18. Question

Find the area of the region bounded by by y = √x and x = 2y + 3 in the first quadrant and x - axis.

Answer

To find an area in the first quadrant enclosed by the x - axis

y = √x ...(i)

x = 2y + 3 ...(ii)

On solving the equation (i) and (ii),

$$y = \frac{x-3}{2}$$

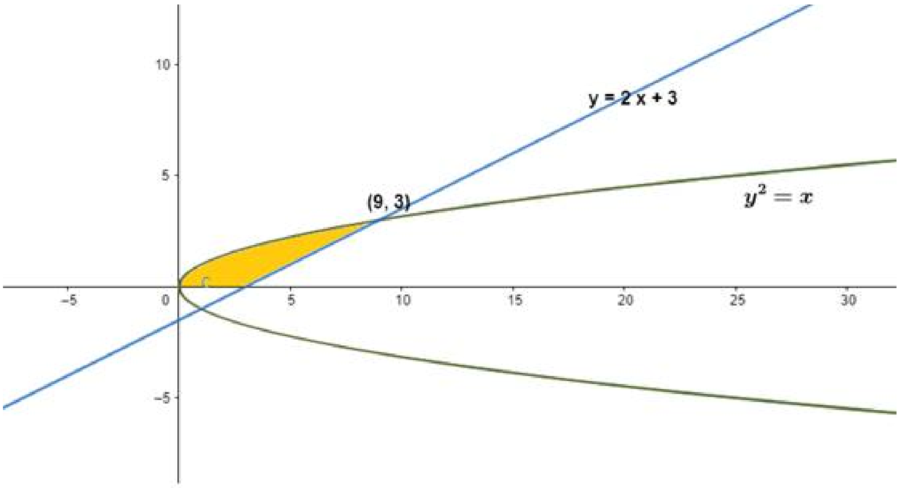
$y^2 = 2y + 3$

Or, $y^2 - 2y - 3 = 0$

Or, (y – 3)(y + 1) = 0

Or, y = 3 or y = – 1

These are shown in the graph below: -



Required area of the bounded region

$$= \int_0^3 \sqrt{x}dx + \int_0^9 \sqrt{x} - \left(\frac{x-3}{2}\right)dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^3 + \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{4} + \frac{3x}{2}\right]_3^9$$

$$= \left[\frac{(3)^{3/2}}{3/2} - 0\right] + \left[\frac{(9)^{3/2}}{3/2} - \frac{(9)^2}{4} + \frac{3(9)}{2} - \frac{(3)^{3/2}}{3/2} + \frac{(3)^2}{4} - \frac{3(3)}{2}\right]$$

$$= 9 \text{ sq. units}$$

The area of the region bounded by by y = √x and x = 2y + 3 is 9 sq. units

19. Question

Find the area common to the circle $x^2 + y^2 = 16a^2$ and the parabola $y^2 = 6ax$.

OR

Find the area of the region $\{(x,y):y^2 \le 6ax\}$ and $\{(x,y):x^2 + y^2 \le 16a^2\}$.

Answer

To find area given equations are

$y^2 = 6ax$ ...(i)

$x^2 + y^2 = 16\ a^2$ ...(ii)

On solving Equation (i) and (ii)

Or $x^2 + (6ax)^2 = 16a^2$

Or $x^2 + (6ax)^2 - 16a^2 = 0$

Or $(x + 8a)(x - 2a) = 0$

Or x = 2a or x = – 8a is not possible solution.

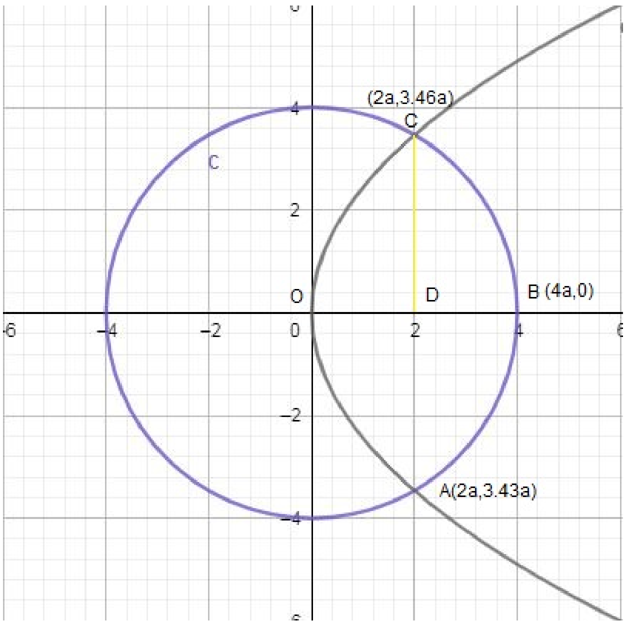
Then $y^2 = 6a(2a) = 12a^2 = \pm 2\sqrt{3}a$

Equation (i) represents a parabola with vertex (0,0) and axis as x - axis.

Equation (ii) represents a with centre (0,0) and meets axes (±4a,0), (0,±4a).

Point of intersection of circle and parabola are (2a,2√3a), (2a, – 2√3a).

These are shown in the graph below: -



Required area = 2[Region ODCO + Region BCDB]

$$= 2\left[\int_0^{2a} y_1 dx + \int_{2a}^{4a} y_2 dx\right]$$

$$= 2\left[\int_0^{2a} \sqrt{6ax}dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2 dx}\right]$$

$$= \left[\sqrt{6a}\left(\frac{2}{3}x\sqrt{x}\right)_0^{2a} + \left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\left(\frac{x}{4a}\right)\right]_{2a}^{4a}\right]$$

$$= \left[\left(\sqrt{6a}.\frac{2}{3}2a\sqrt{2a}\right) + \left[\left(0 + 8a^2.\frac{\pi}{2}\right) - \left(a\sqrt{12a^2} + 8a^2.\frac{\pi}{6}\right)\right]_{2a}^{4a}\right]$$

$$= 2\left[\frac{8\sqrt{3}a^2}{3} + 4a^2\pi - 2\sqrt{3}a^2 - \frac{4}{3}a^2\pi\right]$$

$$= 2\left[\frac{2\sqrt{3}a^2}{3} + \frac{8a^2\pi}{3}\right]$$

$$= \frac{4a^2}{3}(4\pi + \sqrt{3})\text{sq.units}$$

The area common to the circle$x^2 + y^2 = 16\,a^2$ and the parabola $y^2 = 6ax$ is $\frac{4a^2}{3}(4\pi + \sqrt{3})$sq.units

20. Question

Find the area, lying above x - axis and included between the circle circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

Answer

To find area lying above x - axis and included in the circle

$x^2 + y^2 = 8x$ ...(i)

Or $(x - 4)^2 + y^2 = 16$

And $y^2 = 4x$ ...(ii)

On solving the equation (i) and (ii),

$x^2 + y^2 = 8x$

Or $x^2 - 4x = 0$

Or $x(x - 4) = 0$

Or $x = 0$ and $x = 4$

When $x = 0$, $y = 0$

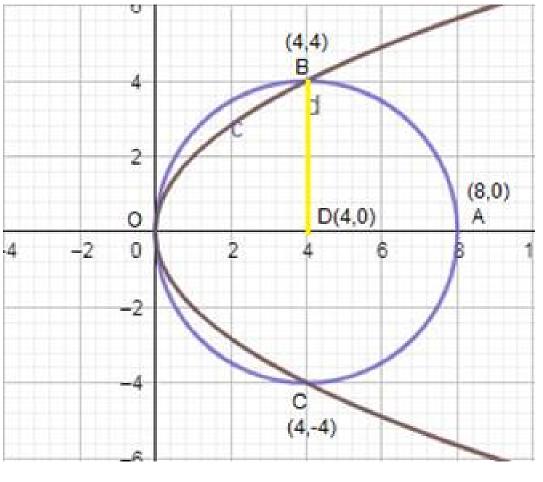
When $x = 4$, $y = \pm 4$

Equation (i) represents a circle with centre (4,0) and meets axes at (0,0) and (8,0).

Equation (ii) represent a parabola with vertex (0,0) and axis as x - axis.

They intersect at (4, – 4) and (4,4).

These are shown in the graph below: -



Required area = Region OABO

Required area = Region ODBO + Region DABD ...(1)

Region ODBO = $\int_0^4 2\sqrt{x}\,dx$

$= 2\left(\frac{2}{3}x\sqrt{x}\right)$

Region OBDO = 32/3 sq. units ...(2)

Region DABD = $\int_4^8 \sqrt{16 - (x-4)^2}\,dx$

$= \left[\frac{(x-4)}{2}\sqrt{16-(x-4)^2} + \frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_4^8$

$= \left[\left(0 + 8.\frac{\pi}{2}\right) - (0 + 0)\right]_4^8$

Region DABD = 4π sq. units ...(3)

Using (1),(2) and (3), We get

Required area = $\left(\frac{32}{3} + 4\pi\right)$

= 4 $\left(\pi + \frac{8}{3}\right)$ sq. units

The area lying above the x - axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$ is 4 $\left(\pi + \frac{8}{3}\right)$ sq. Units

21. Question

Find the area enclosed by the parabolas $y = 5x^2$ and $y = 2x^2 + 9$.

Answer

To find the area enclosed by

$y = 5x^2$ ...(i)

and $y = 2x^2 + 9$ ...(ii)

On solving the equation (i) and (ii),

$5x^2 = 2x^2 + 9$

Or $3x^2 = 9$

Or $x = \pm\sqrt{3}$

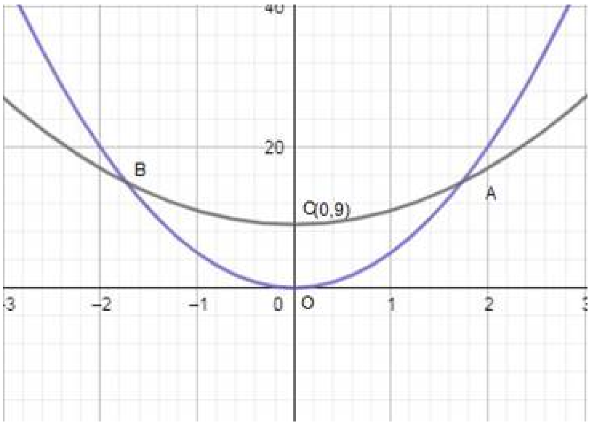
Or $y = 15$

Equation (i) represents a parabola with vertex O (0, 0) and axis as y - axis .

Equation (ii) represents a parabola with vertex C (0, 9) and axis as the y - axis.

Points of intersection of parabolas are A (√3, 15) and B(– √3,15)

These are shown in the graph below: -



Required area = Region AOBCA

= 2(Region AOCA)

$= 2\int_0^{\sqrt{3}} (2x^2 + 9 - 5x^2)dx$

$= 2\int_0^{\sqrt{3}} (9 - 3x^2)dx$

$= [9x - x^3]_0^{\sqrt{3}}$

$= 2\left[\left(9\sqrt{3} - 3\sqrt{3}\right) - (0)\right]$

$= 12\sqrt{3}\text{sq. units}$

The area enclosed by the parabolas $y = 5x^2$ and $y = 2x^2 + 9$ is $12\sqrt{3}$ sq. units

22. Question

Prove that the area common to the two parabolas $y = 2x^2$ and $y = x^2 + 4$ is 32/3 sq. Units.

Answer

To find the area enclosed by,

$y = 2x^2$ ...(i)

And $y = x^2 + 4$ ...(ii)

On solving the equation (i) and (ii),

$2x^2 = x^2 + 4$

Or $x^2 = 4$

Or $x = \pm 2$

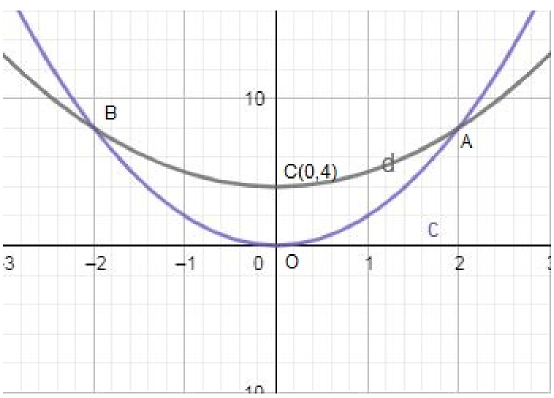
$\therefore y = 8$

Equation (1) represents a parabola with vertex (0,0) and axis as y - axis.

Equation (2) represents a parabola with vertex (0,4) and axis as the y - axis.

Points of intersection of parabolas are A(2,8) and B(– 2,8).

These are shown in the graph below: -



Required area = Region AOBCA

= 2(Region AOCA)

$= 2\int_0^2 (x^2 + 4 - 2x^2)dx$

$= 2\int_0^2 (4 - x^2)dx$

$= 2\left[4x - \frac{x^3}{3}\right]_0^2$

$= 2\left[\left(8 - \frac{8}{3}\right) - (0)\right]$

$= \frac{32}{3}\text{sq. units}$

Hence, proved that the area common to the two parabolas y = $2x^2$ and y = x^2 + 4 is 32/3 sq. Units.

23. Question

Using integration, find the area of the region bounded by the triangle whose vertices are

(i) (– 1, 2), (1, 5) and (3, 4)

(ii) (– 2, 1), (0, 4) and (2, 3)

Answer

Equation of side AB,

$\frac{x + 1}{1 + 1} = \frac{y - 2}{5 - 2}$

Or $\frac{x+1}{2} = \frac{y-2}{3}$

Or 3x + 3 = 2y – 4

Or 2y – 3x = 7

$\therefore y = \frac{3x+7}{2}$ ...(i)

Similarly, the equation of side BC,

$\frac{x - 1}{3 - 1} = \frac{y - 5}{4 - 5}$

Or $\frac{x-1}{2} = \frac{y-5}{-1}$

Or – x + 1 = 2y – 10

Or 2y = 11 – x

$\therefore y = \frac{11-x}{2}$ (ii)

And equation of side AC,

$\frac{x + 1}{3 + 1} = \frac{y - 2}{4 - 2}$

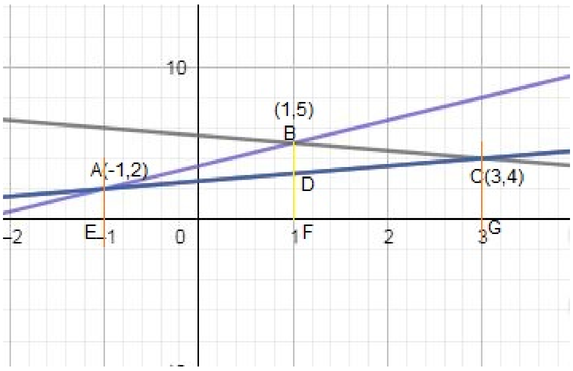
Or $\frac{x+1}{4} = \frac{y-2}{2}$

Or $\frac{x+1}{2} = \frac{y-2}{1}$

Or x + 1 = 2y – 4

$\therefore y = \frac{5+x}{2}$ ...(iii)

These are shown in the graph below: -



Area of required region

= Area of EABFE + Area of BFGCB – Area of AEGCA

$$= \int_{-1}^{1} y_{AB} dx + \int_{1}^{3} y_{BC} dx - \int_{-1}^{3} y_{AC} dx$$

$$= \int_{-1}^{1} \frac{3x + 7}{2} dx + \int_{1}^{3} \frac{11 - x}{2} dx - \int_{-1}^{3} \frac{5 + x}{2} dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} 7x\right]_{-1}^{1} + \frac{1}{2}\left[11x - \frac{x^2}{2}\right]_{1}^{3} - \frac{1}{2}\left[5x + \frac{x^2}{2}\right]_{-1}^{3}$$

$$-\frac{1}{2}\left[5(3-(-1)) + \frac{(3)^2 - 1^2}{2}\right]$$

$$= \frac{1}{2}[0 + 14] + \frac{1}{2}[22 - 4] - \frac{1}{2}[20 + 4]$$

$$= 7 + \frac{1}{2}x18 - \frac{1}{2}x24$$

$$= 7 + 9 - 12$$

= 4 sq. units

The area of the region bounded by the triangle is 4 sq. units.

24. Question

Find the area of the region bounded by y = √x and y = x.

Answer

To find the area bounded by

y = √x ...(i)

y = x ...(ii)

On solving the equation (i) and (ii),

Or x^2 = x

Or, x(x – 1) = 0

Or, x = 0 or x = 1

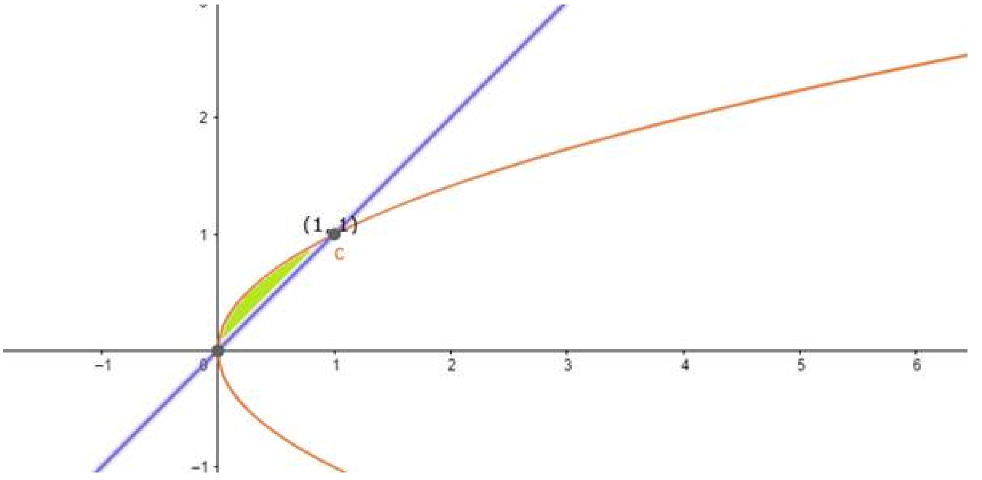
Then y = 0 or y = 1

Equation (i) represent a parabola with vertex (0,0) and axis as x - axis

Equation (ii) represents a line passing through origin.

Points of intersection are (0,0) and (1,1).

These are shown in the graph below: -



Area of bounded region

$= \int_0^1 \sqrt{x} - x dx$

$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$

$= \left[\frac{2}{3} - \frac{1}{2}\right]$

$= \frac{1}{6} \text{sq. units}$

The area of the region bounded by y = √x and y = x is $\frac{1}{6}$ sq. units

25. Question

Find the area of the region in the first quadrant enclosed by the x - axis, the line y = √3x and the circle x^2 + y^2 = 16.

Answer

To find the area enclosed by

y = √3x ...(i)

$x^2 + y^2 = 16$ ...(ii)

On solving the equation (i) and (ii),

Or $x^2 + (\sqrt{3x})^2 = 16$

Or $4x^2 = 16$

Or $x^2 = 4$

Or x = ± 2

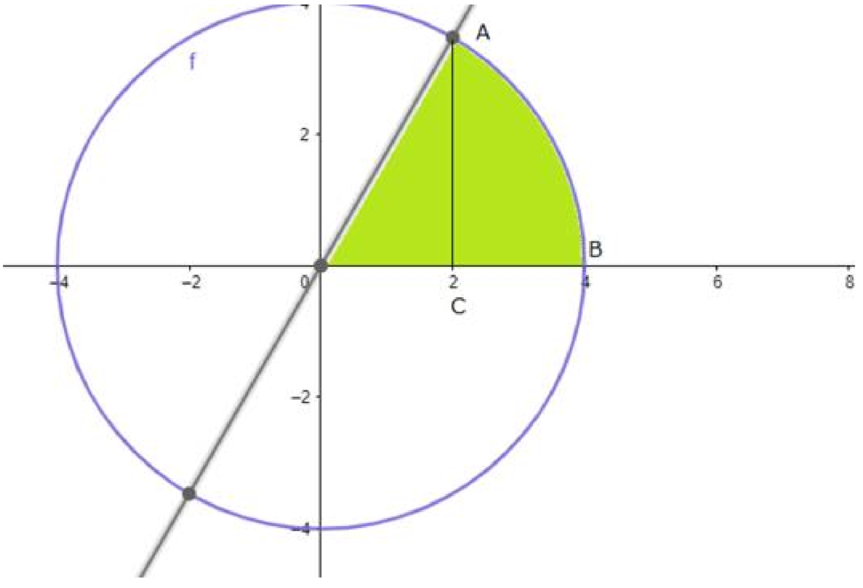
∴ Y = $\pm 2\sqrt{3}$

Equation (i) represents a parabola with vertex (0,0) and axis as x - axis.

Equation (ii) represent axis a circle with centre (4,0) and meets axes at (0,0) and (4,0).

They intersect at A (2, $2\sqrt{3}$) and C (– 2, – $2\sqrt{3}$).

These are shown in the graph below: -



Area of the region OAB = Area OAC + Area ACB

$= \int_0^2 \sqrt{3}x dx + \int_2^4 \sqrt{16 - x^2} dx$

$= \left(\frac{\sqrt{3}x^2}{2}\right)_0^2 + \frac{1}{2}\left[x\sqrt{16 - x^2} + 16\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4$

$= \left(\frac{\sqrt{3}x4}{2}\right) + \frac{1}{2}\left[16\sin^{-1}\left(\frac{4}{4}\right)\right] - \frac{1}{2}\left[4\sqrt{16 - 12} + 16\sin^{-1}\left(\frac{2}{4}\right)\right]$

$= 2\sqrt{3} + \frac{1}{2}\left[16x\frac{\pi}{2}\right] - \frac{1}{2}\left[4\sqrt{3} + 16\sin^{-1}\left(\frac{1}{2}\right)\right]$

$= 2\sqrt{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}$

$= 4\pi - \frac{4\pi}{3}$

$= \frac{8\pi}{3}$ sq. units

The area of the region in the first quadrant enclosed by x - axis, the line y = √3x and the circle $x^2 + y^2 = 16$ is $\frac{8\pi}{3}$ sq. units

26. Question

Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line x – y – 1 = 0.

Answer

To find area bounded by

$y^2 = 2x + 1$ ...(i)

X – y – 1 = 0. ...(ii)

On solving the equation (i) and (ii),

X – y = 1

Or $y^2 = 2(y - 1) + 1$

Or $y^2 = 2y - 1$

Or (y + 1)(y – 3) = 0

Or y = 3 or – 1

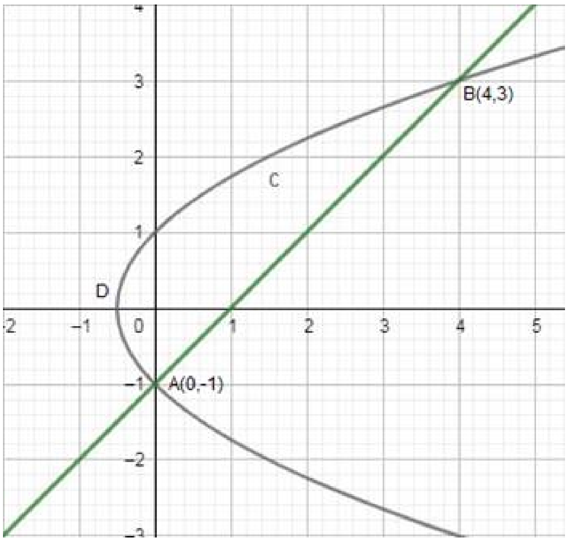
∴ x = 4,0

Equation (i) is a parabola with vertex $\left(-\frac{1}{2}, 0\right)$ and passes through (0, 1), A (0, – 1)

Equation (ii) is a line passing through (1, 0) and (0, – 1).

Points of intersection of parabola and line are B (4, 3) and A (0, – 1)

These are shown in the graph below: -



Required area = Region ABCDA

$= \int_{-1}^{3}\left(1 + y - \frac{y^2 - 1}{2}\right) dy$

$= \frac{1}{2}\int_{-1}^{3}(2 + 2y - y^2 + 1) dy$

$= \frac{1}{2}\int_{-1}^{3}(3 + 2y - y^2) dy$

$= \frac{1}{2}\left[3y + y^2 - \frac{y^3}{3}\right]_{-1}^{3}$

$= \frac{1}{2}\left[(9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3}\right)\right]$

$= \frac{1}{2}\left[9 + \frac{5}{3}\right]$

$= \frac{32}{6}$ sq. units

Area of the region bounded by the parabola $y^2 = 2x + 1$ and the line x – y – 1 = 0is $\frac{32}{6}$ sq. units.

27. Question

Find the area of the region bounded by the curves y = x – 1 and $(y - 1)^2 = 4(x + 1)$.

Answer

To find region bounded by curves

y = x – 1 ...(i)

$(y - 1)^2 = 4(x + 1)$ ...(ii)

On solving the equation (i) and (ii),

Or $(x - 1 - 1)^2 = 4(x + 1)$

Or $(x - 2)^2 = 4(x + 1)$

Or $x^2 + 4 - 4x = 4x + 4$

Or $x^2 - 8x = 0$

Or $x = 0$ or 8

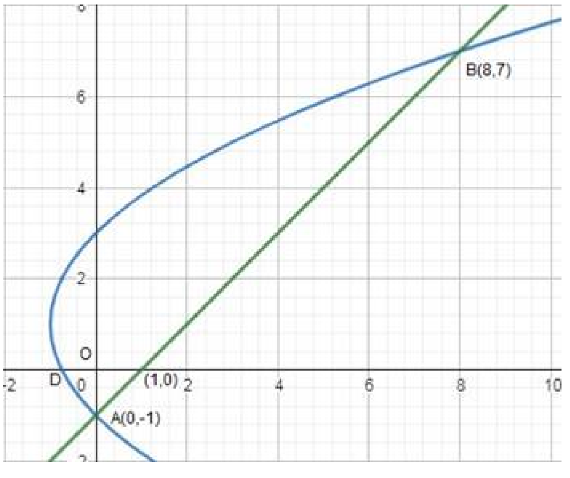
$\therefore y = -1$ or 7

Equation (i) represents a line passing through (1,0) and (0, – 1)

Equation (ii) represents a parabola with vertex (– 1,1) passes through (0,3),(0, – 1),$\left(-\frac{3}{4}, 0\right)$.

Their points of intersection A(0, – 1) and B(8,7).

These are shown in the graph below: –



It slides from y = – 1 to y = 7,

So, required area = Region ABCDA

$$= \int_{-1}^{7} (x_1 - x_2)dy$$

$$= \int_{-1}^{7} \left(y + 1 - \frac{(y + 1)^2}{4} + 1\right) dy$$

$$= \frac{1}{4}\int_{-1}^{7} (4y + 4 - y^2 - 1 + 2y + 4)dy$$

$$= \frac{1}{4}\left(3y^2 + 7y - \frac{y^3}{3}\right)_{-1}^{7}$$

$$= \frac{1}{4}\left[\left(147 + 49 - \frac{343}{3}\right) - \left(3 - 7 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{245}{3} + \frac{11}{3}\right]$$

$$= \frac{64}{3} \text{ sq. units}$$

The area of the region bounded by the curves $y = x - 1$ and $(y - 1)^2 = 4(x + 1)$ is $\frac{64}{3}$ sq. units

28. Question

Find the area enclosed by the curve y = – x^2 and the straight line x + y + 2 = 0

Answer

To find region enclosed by

y = – x^2 ...(i)

x + y + 2 = 0 ...(ii)

On solving the equation (i) and (ii),

x – x^2 + 2 = 0

Or x = 2 or – 1

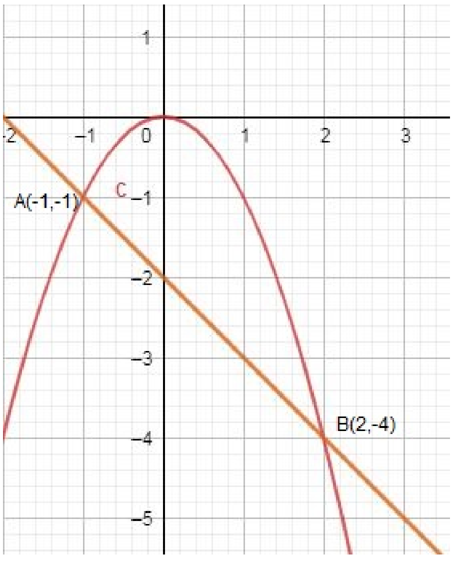
∴y = – 4, – 1

Equation (i) represents a parabola opening towards the negative y - axis.

Equation (ii) represents a line passing through (– 2,0) and (0, – 2).

Their points of intersection A(– 1, – 1) and B(2, – 4).

These are shown in the graph below: -



Area of the bounded region

$$= \int_{-1}^{2} -x^2-(-2-x)dx$$

$$= \left[-\frac{x}{3} + 2x + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \left[-\frac{8}{3} + 6\right] - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$= \frac{9}{2}\text{sq. units}$$

The area enclosed by the curve y = – x^2 and the straight line x + y + 2 = 0 is $\frac{9}{2}$ sq. units

29. Question

Find the area enclosed by the curve Y = 2 – x^2 and the straight line x + y = 0.

Answer

To find region enclosed by

$Y = 2 - x^2$ ...(i)

And x + y = 0 ...(ii)

On solving the equation (i) and (ii),

$x - x^2 + 2 = 0$

Or, $x^2 - x + 2 = 0$

Or, x = 2 or – 1

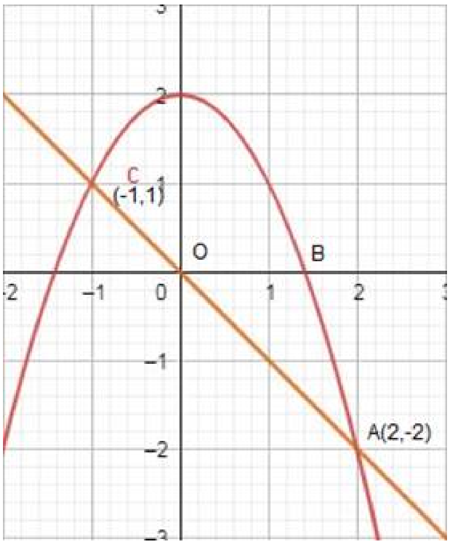
∴ y = – 2 or 1

Equation (i) represents a parabola with vertex (0, 2) and downward, meets axes at (±√2, 0).

Equation (2) represents a line passing through (0, 0) and (2, – 2).

The points of intersection are A (2, – 2) and C (– 1,1).

These are shown in the graph below: -



Required area = Region ABPCOA

$$= \int_{-1}^{2} (2 - x^2 + x)dx$$
$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{3}\right]_{-1}^{2}$$
$$= \left[\left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)\right]$$
$$= \left[\frac{10}{3} + \frac{7}{6}\right]$$
$$= \frac{27}{6}$$
$$= \frac{9}{2}\text{sq. units}$$

The area enclosed by the curve $Y = 2 - x^2$ and the straight line y + x = 0 is $\frac{9}{2}$ sq. units

30. Question

Using the method of integration, find the area of the region bounded by the following line 3x – y – 3 = 0, 2x + y – 12 = 0, x – 2y – 1 = 0.

Answer

To find region enclosed by

3x – y – 3 = 0 ...(i)

2x + y – 12 = 0 ...(ii)

x – 2y – 1 = 0 ...(iii)

Solving (i) and (ii), we get,

5x – 15 = 0

Or x = 3

∴y = 6

The points of intersection of (i) and (ii) is B (3,6)

Solving (i) and (iii), we get,

5x = 5

Or x = 1

∴y = 0

The points of intersection of (i) and (iii) is A (1,0)

Solving (ii) and (iii), we get,

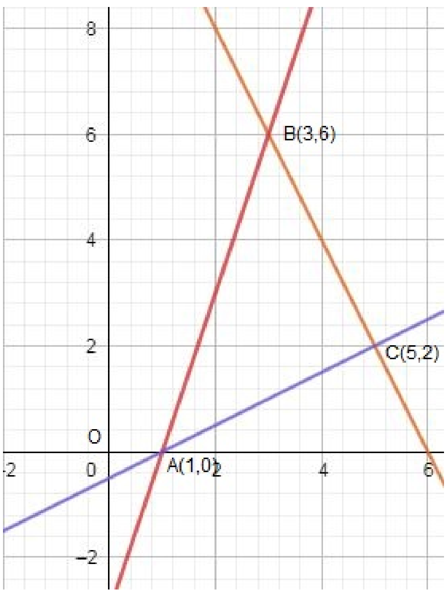
5x = 25

Or x = 5

∴y = 2

The points of intersection of (ii) and (iii) is C (5,2) .

These are shown in the graph below: -



Area of the bounded region

$$= \int_0^3 3x - 3 - \left(\frac{x-1}{2}\right) dx + \int_3^5 12 - 2x - \left(\frac{x-1}{2}\right) dx$$

$$= \left[\frac{3x^2}{2} - 3x - \frac{x^2}{4} + \frac{1}{2}x\right]_0^3 + \left[12x - 2\frac{x^2}{2} - \frac{x^2}{4} + \frac{1}{2}\right]_3^5$$

$$= \left[\frac{27}{2} - 9\frac{9}{4} + \frac{3}{3}\right] + \left[60 - 25 - \frac{25}{4} + \frac{5}{2} - 36 + 9 + \frac{9}{4} - \frac{3}{2}\right]$$

= 11 sq. units

The area of the region bounded by the following line 3x – y 3 = 0, 2x + y – 12 = 0, x – 2y – 1 = 0 is
11 sq. units

31. Question

Sketch the region bounded by the curves $y = x^2 + 2$, y = x, x = 0 and x = 1. Also, find the area of the region.

Answer

To find area bounded x = 0, x = 1

And y = x ...(i)

$y = x^2 + 2$ ...(ii)

Putting x = 1 in equation (ii) we get,

Y = 1 + 2 = 3

Putting x = 1 in equation (i) we get,

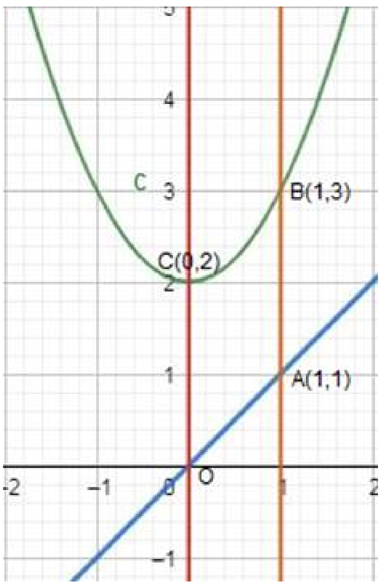
Y = 1

So the point of intersection B (1,3), A(1,1)

Equation (i) is a line passing through (1,1) and (0, 0)

Equation (2) is a parabola upward with vertex at (0, 2).

These are shown in the graph below: -



Required area = Region OABCO

$$= \int_0^1 (y_1 - y_2)dx$$

$$= \int_0^1 (x^2 + 2 - x)dx$$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{3}\right]_0^1$$

$$= \left[\left(\frac{1}{3} + 2 - \frac{1}{2}\right) - (0)\right]$$

$$= \left(\frac{2 + 12 - 3}{6}\right)$$

$$= \frac{11}{6} \text{sq. units}$$

The area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 1$ is $\frac{11}{6}$ sq. units

32. Question

Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

Answer

To find area bounded by

$X = y^2$ ...(i)

And

$X = 3 - 2y^2$ ...(ii)

On solving the equation (i) and (ii),

$y^2 = 3 - 2y^2$

Or $3y^2 = 3$

Or $y = \pm 1$

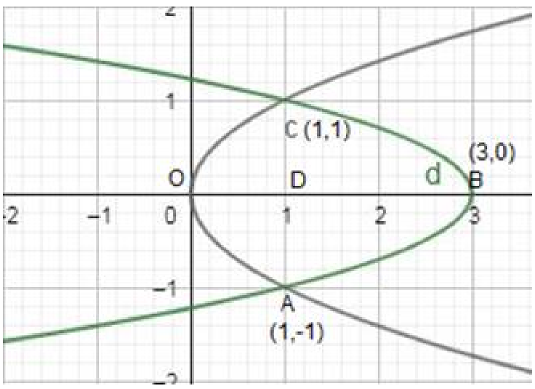
When y = 1 then x = 1 and when y = – 1 then x = 1

Equation (i) represents an upward parabola with vertex (0, 0) and axis – y.

Equation (ii) represents a parabola with vertex (3, 0) and axis as x – axis.

They intersect at A (1, – 1) and C (1, 1)

These are shown in the graph below: -



Required area = Region OABCO

= 2 Region OBCO

= 2[Region ODCO + Region BDCB]

$$= 2\left[\int_0^1 y_1 dx + \int_1^3 y_2 dx\right]$$

$$= 2\left[\int_0^1 \sqrt{x}dx + \int_1^3 \sqrt{\frac{3-x}{2}}dx\right]$$

$$= 2\left[\left(\frac{2}{3}x\sqrt{x}\right)_0^1 + \left(\frac{2}{3}.\left(\frac{3-x}{2}\right)\sqrt{\frac{3-x}{2}}.(-2)\right)_1^3\right]$$

$$= 2\left[\left(\frac{2}{3} - 0\right) + \left\{(0) - \left(\frac{2}{3}.1.1.(-2)\right)\right]\right]$$

$= 2\left[\frac{2}{3} + \frac{4}{3}\right]$

$= 4 \text{ sq. units}$

The area bounded by the curves x = y^2 and x = 3 – 2 y^2 is 4 sq. units

33. Question

Using integration, find the area of the triangle ABC coordinates of whose vertices are A (4, 1), B (6, 6) and C (8, 4).

Answer

To find area of ΔABC with A (4,1), B(6,6) and C(8,4)

Equation of AB,

$y - y_1 = 1\left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

$y - 1 = \left(\frac{6-1}{6-4}\right)(x - 4)$

$y - 1 = \frac{5}{2}x - 10$

$y = \frac{5}{2}x - 9$ ...(i)

Equation of BC,

$y - 6 = \left(\frac{4-6}{8-6}\right)(x - 6)$

y – 6 = – 1(x – 6)

y = – x + 12 ...(ii)

Equation of AC,

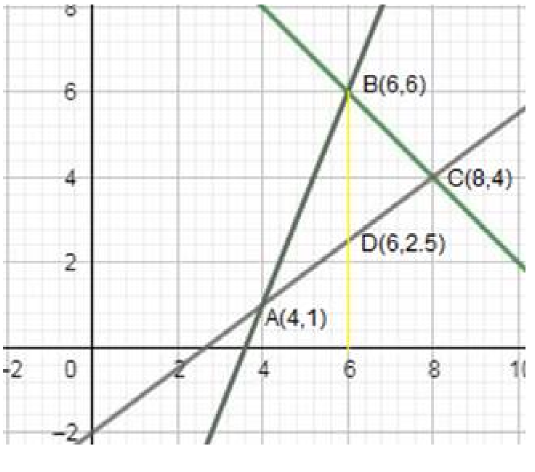
$y - 1 = \left(\frac{4-1}{8-4}\right)(x - 4)$

$y - 1 = \frac{3}{4}(x - 4)$

$\Rightarrow y = \frac{3}{4}x - 3 + 1$

$y = \frac{3}{4}x - 2 \text{ ... (iii)}$

These are shown in the graph below: -



Clearly, Area of ΔABC = Area ADB + Area BDC

Area of ADB,

$= \int_4^6 (y_2 - y_1)dx$

$= \int_4^6 \left[\left(\frac{5x}{2} - 9\right) - \left(\frac{3}{4}x - 2\right)\right] dx$

$= \int_4^6 \left(\frac{5x}{2} - 9 - \frac{3}{4}x + 2\right) dx$

$= \int_4^6 \left(\frac{7x}{4} - 7\right) dx$

$= \left(\frac{7x^2}{4x2} - 7x\right)_4^6$

$= \left(\frac{7x36}{8} - 7x6\right) - \left(\frac{7x16}{8} - 7x4\right)$

$= \left(\frac{63}{2} - 42 - 14 + 28\right)$

$= \left(\frac{63}{2} - 28\right)$

Similarly, $\text{Area of BDC} = \int_6^8 (y_4 - y_3)dx$

$= \int_6^8 \left[(-x + 12) - \left(\frac{3}{4}x - 2\right)\right] dx$

$= \int_6^8 \left[-\frac{7x}{4} + 14\right] dx$

$= \left[-\frac{7x^2}{8} + 14x\right]_6^8$

$= \left[-\frac{7x64}{8} + 14x8\right] - \left[-\frac{7x36}{8} + 14x6\right]$

$= \left[-56 + 112 + \frac{63}{2} - 84\right]$

$= \left(\frac{63}{2} - 28\right)$

Thus, Area ABC = Area ADB + Area BDC

$= \left(\frac{63}{2} - 28\right) + \left(\frac{63}{2} - 28\right)$

= 64 – 56

= 7 sq. units

The area of the triangle ABC coordinates of whose vertices are A (4, 1), B (6, 6) and C (8, 4) is 7 sq. Units

34. Question

Using integration find the area of the region $\{(x,y)| |x - 1| \le y \le \sqrt{5 - x^2}\}$.

Answer

To find area of region

$\{(x,y)| |x - 1| \le y \le \sqrt{5 - x^2}\}$

$|x - 1| = y$

$$\Rightarrow y = \begin{cases} 1 - x, \text{if} x < 1 \ \ldots \text{(i)} \\ x - 1, \text{if} x \ge 1 \ \ldots \text{(ii)} \end{cases}$$

And $x^2 + y^2 = 5$ ...(iii)

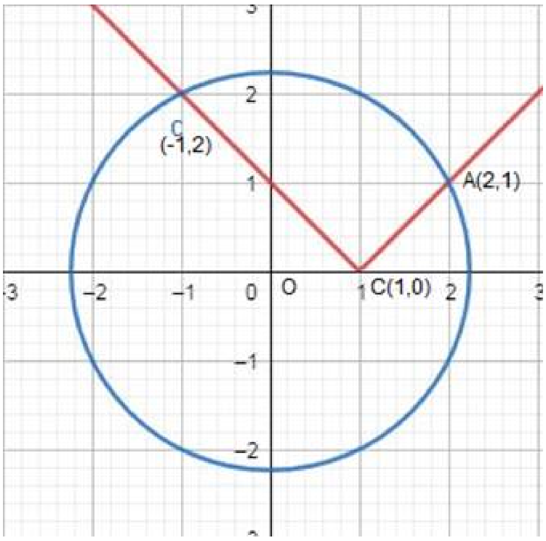
$|x - 1| \le y \le \sqrt{5 - x^2}$

$|x - 1| = \sqrt{5 - x^2}$

X = 2, – 1

Equation (i) and (ii) represent straight lines and equation (iii) is a circle with centre (0,0), meets axes at $(\pm\sqrt{5},0)$ and $(0,\pm\sqrt{5})$.

These are shown in the graph below: -



Required area = Region BCDB + Region CADC

$$A = \int_{-1}^{1}(y_1 - y_2)dx + \int_{1}^{2}(y_1 - y_2)dx$$

$$= \int_{-1}^{1}\left[\sqrt{5 - x^2} - 1 + x\right]dx + \int_{1}^{2}\left(\sqrt{5 - x^2} - x + 1\right)dx$$

$$= \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}x + \frac{x^2}{2}\right]_{-1}^{1}$$

$$+ \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}x - \frac{x^2}{2} + x\right]_{1}^{2}$$

$$= \left[\left(\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - 1 + \frac{1}{2}\right) - \left(\frac{1}{2}.2 - \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + 1 + \frac{1}{2}\right)\right]$$

$$+ \left[\left(1.1. + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 2 + 2\right)\right.$$

$$\left. - \left(\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2} + 1\right)\right]$$

$$= \left[1 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2} + 1\frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{3}{2}\right]$$

$$+ \left[1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - \frac{1}{2} - 1 - \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2}\right]$$

$$= 5\sin^{-1}\frac{1}{\sqrt{5}} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2}$$

$$A = \left[\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right) - \frac{1}{2}\right] \text{sq. units.}$$

The area of the region $\{(x,y)|x - 1|\le y\le\sqrt{s - x^2}\}$ is $\left[\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right) - \frac{1}{2}\right]$ sq. units.

35. Question

Find the area of the region bounded by y – |x – 1| and y = 1.

Answer

To find are bounded by

Y = |x – 1|

y = 1

$$\Rightarrow y = \begin{cases} x - 1, \text{if} x \ge 0 \ \dots (i) \\ 1 - x, \text{if} x < 0 \ \dots (ii) \end{cases}$$

y = x – 1

or, 1 = x – 1

or, x – 2 = 0

or, x = 2

C(2,1) is point of intersection of y = x – 1 and y = 1.

y = 1 – x

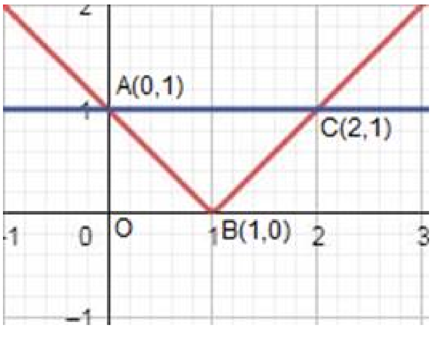
1 = 1 – x

X = 0

A(0,1) is point of intersection of y = 1 – x and y = 1.

Points of intersection are A (0, 1) and C(2,1)

These are shown in the graph below: -



Required area = Region ABCA

= Region ABDA + Region BCDB

$$= \int_0^1 (y_1 - y_2)dx + \int_1^2 (y_1 - y_3)dx$$

$$= \int_0^1 (1 - 1 + x)dx + \int_1^2 (1 - x + 1)dx$$

$$= \int_0^1 x dx + \int_1^2 (2 - x)dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left[(4 - 2) - \left(2 - \frac{1}{2}\right)\right]$$

$= \frac{1}{2} + \left(2 - 2 + \frac{1}{2}\right)$

= 1 sq. units

The area of the region bounded by y = |x – 1| and y = 1 is 1 sq. unit

36. Question

Find the area of the region bounded by y = x and circle $x^2 + y^2 = 32$ in the 1st quadrant.

Answer

To find area of in first quadrant enclose by the circle

$X^2 + y^2 = 32$ ...(i)

And y = x ...(ii)

Solving these two equations, we get

Or $2X^2 = 32$

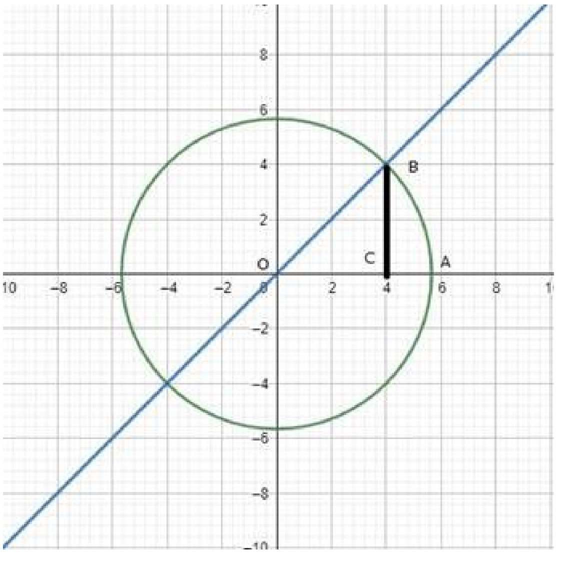
Or $X^2 = 16$

Or $x = \pm 4$

$\therefore y = \pm 4$

Equation (i) is a circle with centre (0, 0) and meets axes at A (±4√2, 0), (0,±4√2). And y = x is a line passes through (0, 0) and intersect circle at B (4, 4).

These are shown in the graph below:



Region OABO = Region OCBO + Region CABC

$$= \int_0^4 y_1 dx + \int_4^4 y_2 dx$$

$$= \int_0^4 x dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2} dx$$

$$= \left(\frac{x^2}{2}\right)_0^4 + \left[\frac{x}{2}\sqrt{32 - x^2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= (8 - 0) + \left[\left(0 + 16.\frac{\pi}{2}\right) - \left(8 + 16.\frac{\pi}{4}\right)\right]$$

$$= 8 + 8\pi - 8 - 4\pi$$

$= 4\pi \text{sq. units}$

The area of the region bounded by y = x and circle $X^2 + y^2 = 32$ is $4\pi \text{sq. units}$

37. Question

Find the area of the circle $x^2 + y^2 = 16$ which is exterior the parabola $y^2 = 6x$.

Answer

The given equations are

$x^2 + y^2 = 16$ ...(i)

And $y^2 = 6x$ ...(ii)

On solving the equation (i) and (ii),

Or $x^2 + 6x = 16$

Or $x^2 + 6x - 16 = 0$

Or $(x + 8)(x - 2) = 0$

Or x = 2 or – 8 is not possible solution

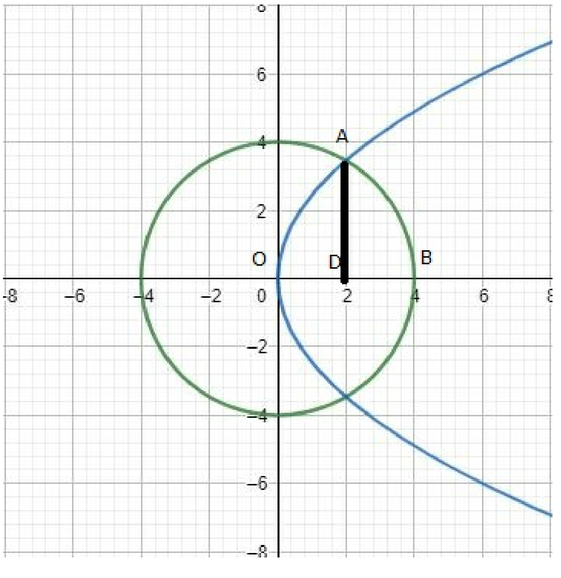
∴ When x = 2, $y = \pm\sqrt{6 \times 2} = \pm\sqrt{12} = \pm 2\sqrt{3}$

Equation (i) is a circle with centre (0, 0) and meets axes at (±4,0), (0,±4)

Equation (ii) represents a parabola with axis as x - axis.

Points of intersection are A $(2, 2\sqrt{3})$ and C $(2, -2\sqrt{3})$

These are shown in the graph below:



Area bounded by the circle and parabola

= 2[Area (OADO) + Area (ADBA)]

$$= 2\left[\int_3^2 \sqrt{16x}dx + \int_2^4 \sqrt{16 - x^2}dx\right]$$

$$= 2\left[\sqrt{6}\left\{\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\}_0^2\right] + 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 2\sqrt{6}x\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2 + 2\left[8.\frac{\pi}{2} - \sqrt{16 - 4} - 8\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$= \frac{4\sqrt{6}}{3}(2\sqrt{2}) + 2\left[4\pi - \sqrt{12} - 8\frac{\pi}{6}\right]$

$= \frac{16\sqrt{3}}{3} + 8\pi - 4\sqrt{3} - \frac{8}{3}\pi$

$= \frac{4}{3}[4\sqrt{3} + 6\pi - 3\sqrt{3} - 2\pi]$

$= \frac{4}{3}[\sqrt{3} + \sqrt{4}x]$

$= \frac{4}{3}[4\pi + \sqrt{3}]\text{squareunits}$

Area of circle = π(r) ^{2}

= $\pi(4)^2$

= 16π sq. Units

Thus, required area = $16\pi - \frac{4}{3}[4\pi + \sqrt{3}]$

$= \frac{4}{3}[4x3\pi - 4\pi - \sqrt{3}]$

$= \frac{4}{3}[8\pi - \sqrt{3}]$

$= \left(\frac{32}{3}\pi - \frac{4\sqrt{3}}{3}\right)\text{sq. unit}$

The area of the circle $x^2 + y^2 = 16$ which is exterior the parabola $y^2 = 6x$ is

$\left(\frac{32}{3}\pi - \frac{4\sqrt{3}}{3}\right)$ sq. units

38. Question

Find the area of the region enclosed by the parabola $x^2 = y$ and the line y = x + 2.

Answer

To Find the area of the region enclosed by

$x^2 = y$ ...(i)

y = x + 2 ...(ii)

On solving the equation (i) and (ii),

Or $x^2 - x - 2 = 0$

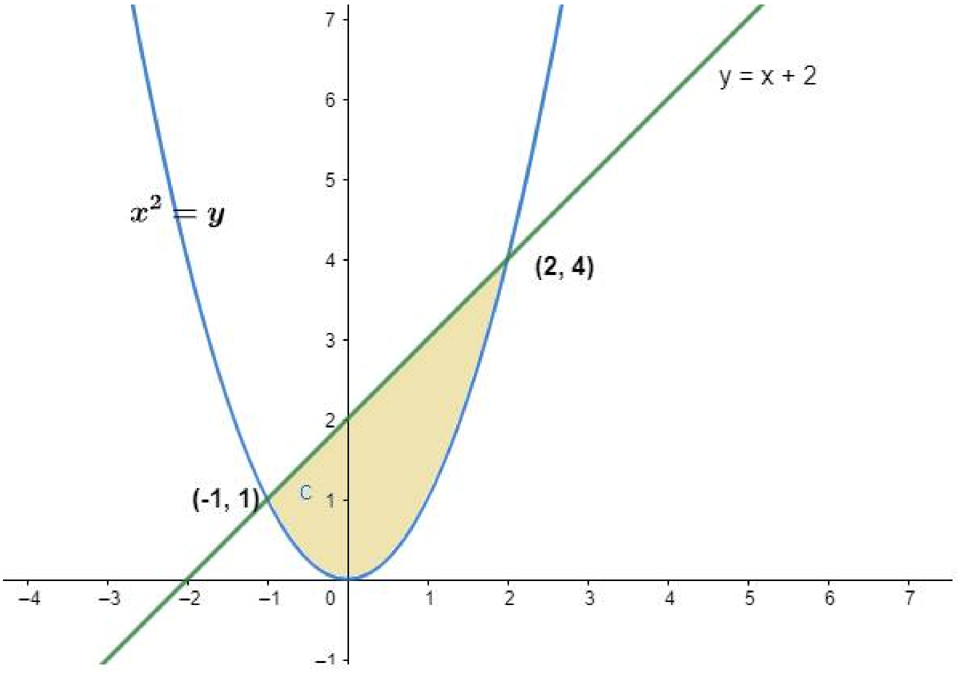
Or (x – 2) (x + 1) = 0

Or x = 2 or x = – 1

∴y = 4 or y = 1

Points of intersection are A $(2,4)$ and C $(-1,1)$

These are shown in the graph below: -



Area of the bounded region

$$= \int_{-1}^{2} x + 2 - x^2 dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{9}{2} \text{sq. units}$$

The area of the region enclosed by the parabola $x^2 = y$ and the line y = x + 2 is $\frac{9}{2}$ sq. units

39. Question

Make a sketch of the region{(x,y): 0 ≤ y ≤ x^2 + 3; 0 ≤ y ≤ 2x + 3; 0 ≤ x ≤ 3} and find its area using integration.

Answer

To find area given equations are

Y = x^2 + 3 ...(i)

Y = 2x + 3 ...(ii)

And x = 3 ...(iii)

Solving the above three equations to get the intersection points,

x^2 + 3 = 2x + 3

Or x^2 – 2x = 0

Or x(x – 2) = 0

And x = 0 or x = 2

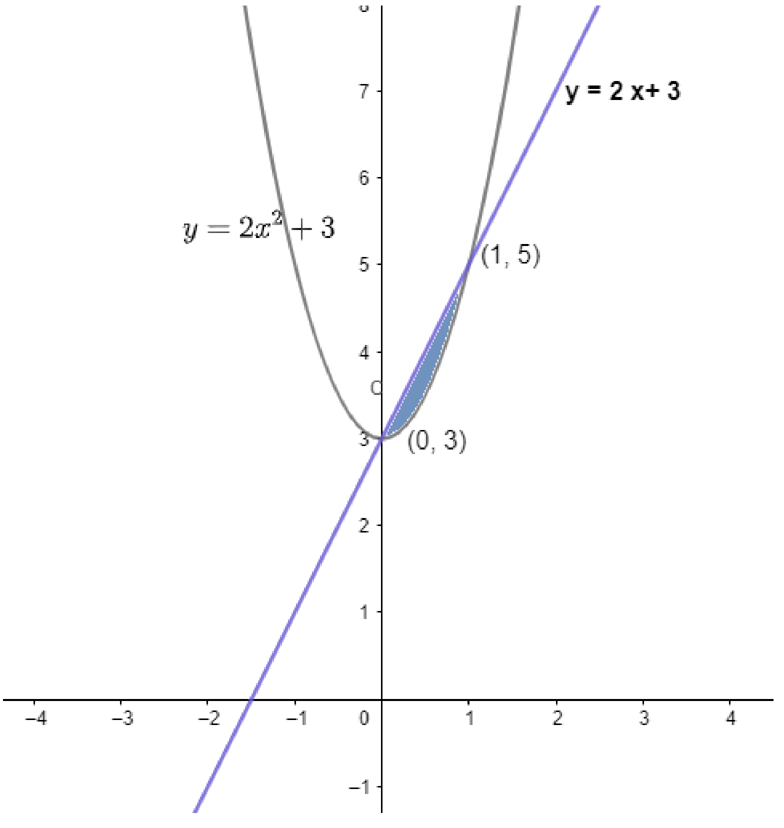
∴ y = 3 or y = 7

Equation (1) represents a parabola with vertex (3, 0) and axis as y – axis.

Equation (2) represents a line a passing through (0, 3) and (– 3/2, 0)

The points of intersection are A (0,3) and B(2,7).

These are shown in the graph below:



Required area =

$= \int_2^3 y_1 dx + \int_0^2 y_2 dx$

$= \int_2^3 (2x + 3)dx + \int_0^2 (x^2 + 3)dx$

$= (x^2 + 3x)_2^3 + \left(\frac{x^3}{3} + x\right)_0^2$

$= [(9 + 9) - (4 + 6)] + \left[\left(\frac{8}{3} + 2\right) - (0)\right]$

$= [18 - 10] + \left[\frac{14}{3}\right]$

$= 8 + \frac{14}{3}$

$= \frac{38}{3}$ sq. units

The area of the region{(x,y): 0≤y≤x^2 + 3; 0 ≤ y ≤ 2x + 3; 0 ≤ x ≤ 3} is $\frac{38}{3}$ sq. units

40. Question

Find the area of the region bounded by the curve y = √1 – x^2, line y = x and the positive x - axis.

Answer

To find the area of the region bounded by

y = √1 – x^2 ...(i)

$X^2 + y^2 = 1$

$X = y$ ...(ii)

On solving the equation (i) and (ii),

Or $X^2 + x^2 = 1$

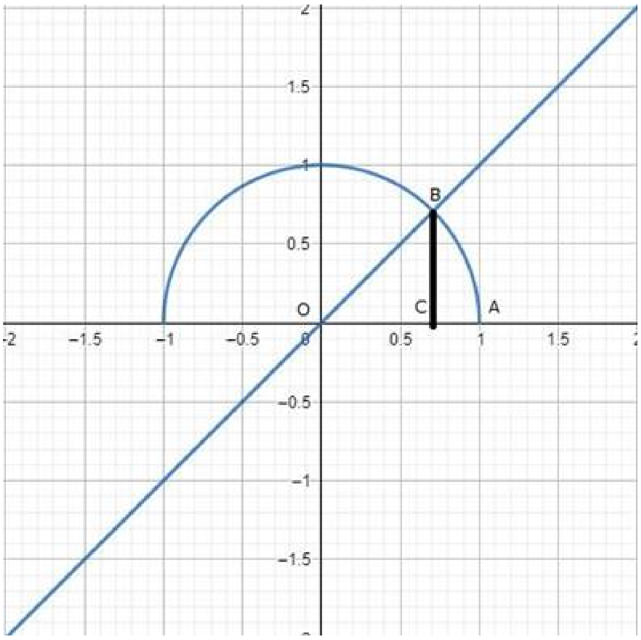
Or $2X^2 = 1$

Or $x = \pm\frac{1}{\sqrt{2}}$

Equation (i) represents a circle (0,0) and meets axes at (±1,0), (0,±1).

Equation (ii) represents a line passing through B $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and they are also points of intersection.

These are shown in the graph below:



Required area = Region OABO

= Region OCBO + Region CABC

$$= \int_0^{\frac{1}{\sqrt{2}}} y_1 dx + \int_{\frac{1}{\sqrt{2}}}^{1} y_2 dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x dx + \int_{\frac{1}{\sqrt{2}}}^{1} \sqrt{1 - x^2 dx}$$

$$= \left[\frac{x^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} + \left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= \left[\frac{1}{4} - 0\right] + \left[\left(0 + \frac{1}{2}.\frac{\pi}{2}\right) - \left(\frac{1}{2\sqrt{2}}.\frac{1}{\sqrt{2}} + \frac{1}{2}.\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{\pi}{4} - \frac{1}{4} - \frac{\pi}{8}$$

$$= \frac{\pi}{8} \text{sq. units}$$

The area of the region bounded by the curve $y = \sqrt{1 - x^2}$, line $y = x$ is $\frac{\pi}{8}$ sq. units

41. Question

Find the area bounded by the lines y = 4x + 5, y = 5 – x and 4y = x + 5.

Answer

To find the area bounded by

Y = 4x + 5 (Say AB) ...(i)

Y = 5 – x (Say BC) (ii)

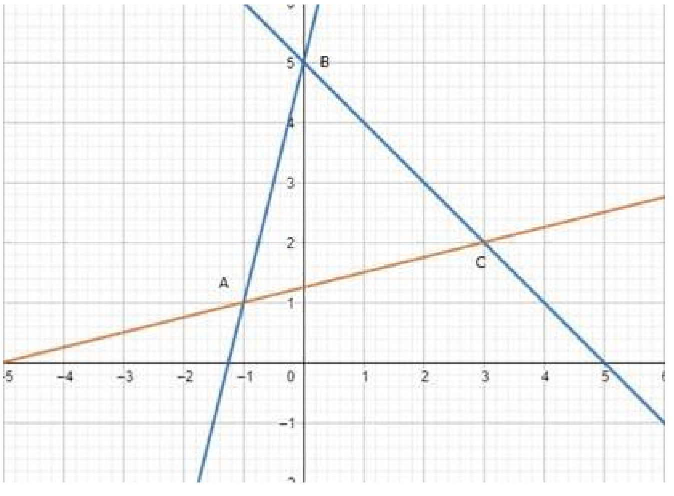
4y = x + 5 (Say AC) ...(iii)

By solving equation (i) and (ii), points of intersection is B (0, 5)

By solving equation (ii) and (iii), points of intersection is C (3, 2)

By solving equation (i) and (iii), we get points of intersection is A (– 1, 1)

These are shown in the graph below:



Required area = area of (ΔABD) + area of (ΔBDC) ...(1)

Area of (ΔABD) = $\int_{-1}^{0}(y_1 - y_3)dx$

$= \int_{-1}^{0}\left(4x + 5 - \frac{x}{4} - \frac{5}{4}\right)dx$

$= \int_{-1}^{0}\left(\frac{15x}{4} + \frac{15}{4}\right)dx$

$= \frac{15}{4}\left(\frac{x^2}{2} + x\right)_{-1}^{0}$

$= \frac{15}{4}\left[(0) - \left(\frac{1}{2} - 1\right)\right]$

$= \frac{15}{4} \times \frac{1}{2}$

$= \frac{15}{8}$ sq. units ...(2)

Area of (ΔBDC) = $\int_{0}^{3}(y_1 - y_2)dx$

$= \int_{0}^{3}\left((5 - x) - \left(\frac{x}{4} + \frac{5}{4}\right)\right)dx$

$= \int_0^3 \left[5 - x - \frac{x}{4} - \frac{5}{4}\right] dx$

$= \int_0^3 \left(\frac{-5x}{4} + \frac{15}{4}\right) dx$

$= \frac{5}{4}\left(3x - \frac{x^2}{2}\right)$

$= \frac{5}{4}\left(9 - \frac{9}{2}\right) = \frac{45}{8} \text{sq. units}$

Area of (ΔBDC) = $\frac{45}{8}\text{sq. units}$ ...(3)

Using equation (1), (2) and (3)

Required area = Area (ΔABD) + Area (ΔBDC)

$= \frac{15}{8} + \frac{45}{8}$

$= \frac{60}{8} = \frac{15}{2}$

Required bounded area of (ΔABC) = Area of (ΔABD) + Area of (ΔBDC) = $\frac{15}{2}\text{sq. units}$

42. Question

Find the area of the region enclosed between the two curves $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$.

Answer

To find area enclosed by

$X^2 + y^2 = 9$ (i)

$(x - 3)^2 + y^2 = 9$ (ii)

On solving equation (i) and (ii) we get,

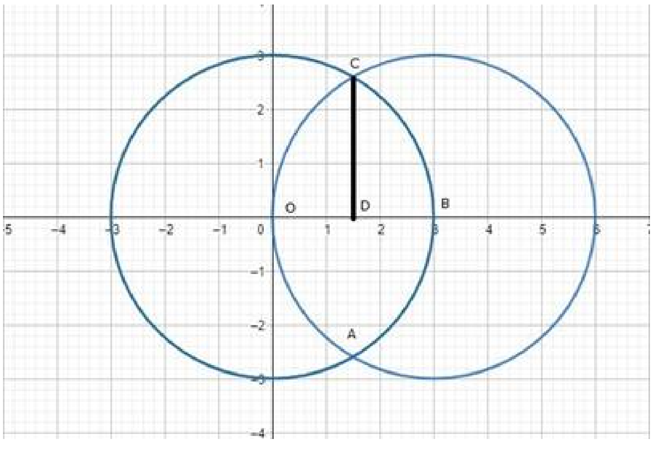
$X = \frac{3}{2}$ and $y = \pm\frac{3\sqrt{3}}{2}$

Equation (i) represents a circle with centre (0,0) and meets axes at (±3,0),(0,±3).

Equation (ii) is a circle with centre (3,0) and meets axe at (0,0), (6,0).

They intersect each other at $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$and $\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$.

These are shown in the graph below:



Required area = Region OABCO

= 2(Region OBCO)

= 2(Region ODCO + Region DBCD)

$$= 2\left[\int_0^{\frac{3}{2}}\sqrt{9-(x-3)^2}dx + \int_{\frac{3}{2}}^{3}\sqrt{9-x^2}dx\right]$$

$$= 2\left[\left\{\frac{(x-3)}{2}\sqrt{9-(x-3)^2} + \frac{9}{2}\sin^{-1}\frac{(x+3)}{3}\right\}_0^{\frac{3}{2}} + \left\{\frac{x}{2}\sqrt{9-x^2} + 9\sin^{-1}\left(\frac{x}{3}\right)\right\}_{\frac{3}{2}}^{3}\right]$$

$$= 2\left[\left\{\left(-\frac{3}{4}\sqrt{9-\frac{9}{4}} + \frac{9}{2}\sin^{-1}\left(-\frac{3}{6}\right)\right) - \left(0 + \frac{9}{2}\sin^{-1}(-1)\right)\right\} + \left\{\left(0 + \frac{9}{2}\sin^{-1}(1)\right) - \left(\frac{3}{4}\sqrt{9-\frac{9}{4}} + \frac{9}{2}\sin^{-1}\left(\frac{1}{2}\right)\right)\right\}\right]$$

$$= 2\left[\left\{-\frac{9\sqrt{3}}{8} - \frac{9}{2}\cdot\frac{\pi}{6} + \frac{9}{2}\cdot\frac{\pi}{2}\right\} + \left\{\frac{9}{2}\cdot\frac{\pi}{2} - \frac{9\sqrt{3}}{8} - \frac{9}{2}\cdot\frac{\pi}{6}\right\}\right]$$

$$= 2\left[-\frac{9\sqrt{3}}{8} - \frac{3\pi}{4} + \frac{9\pi}{4} + \frac{9\pi}{4} - \frac{9\sqrt{3}}{8} - \frac{3\pi}{4}\right]$$

$$= 2\left[\frac{12\pi}{4} - \frac{18\sqrt{3}}{8}\right]$$

$$= \left(6\pi - \frac{9\sqrt{3}}{2}\right)\text{sq. units}$$

The area of the region enclosed between the two curves curves $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$ is $\left(6\pi - \frac{9\sqrt{3}}{2}\right)$ sq. units

43. Question

Find the area of the region $\{(x,y): x^2 + y^2 \le 4, x + y \ge 2\}$

Answer

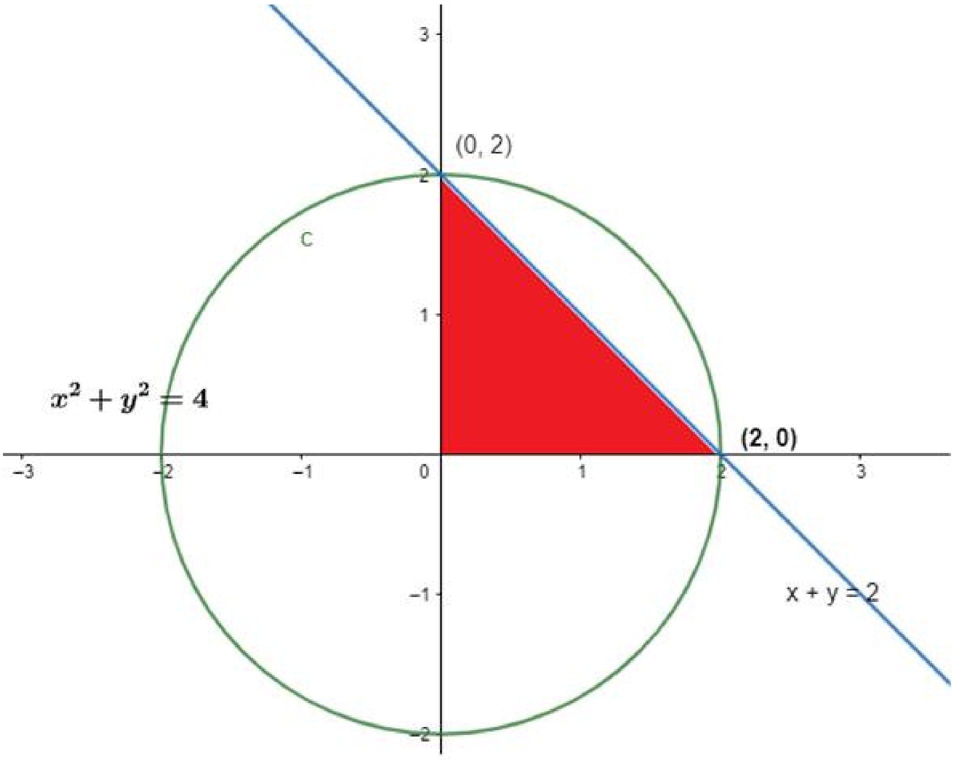
The equation of the given curves are

$X^2 + y^2 = 4$ (i)

$X + y = 2$ (ii)

Clearly $X^2 + y^2 = 4$ represents a circle $X + y = 2$ is the equation of a straight line cutting x and y axes at (0, 2) and (2, 0) respectively.

These are shown in the graph below:



The required area is given by

$$A = \int_0^2 (y_2 - y_1)dx$$

We have $y_1 = 2 - x$ and $y_2 = \sqrt{4-x^2}$

$$A = \int_0^2 \left(\sqrt{4-x^2} - 2 + x\right) dx$$

$$= \int_0^2 \left(\sqrt{4-x^2}\right) dx - 2\int_0^2 dx + \int_0^2 x dx$$

$$= \left[\frac{x\sqrt{4-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - 2(x)_0^2 + \left(\frac{x^2}{2}\right)_0^2$$

$$= \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right) - 4 + 2$$

$$= 2\sin^{-1}(1) - 2$$

$$= 2 \times \frac{\pi}{2} - 2$$

$$= \pi - 2 \text{ sq. units}$$

the area of the region $\{(x,y): x^2 + y^2 \le 4,\ x + y \ge 2\}$ is $\pi - 2$ sq. units

44. Question

Using integration, find the area of the following region $\left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \le 1 \le \frac{x}{3} + \frac{y}{2}\right\}$.

Answer

To find area of region

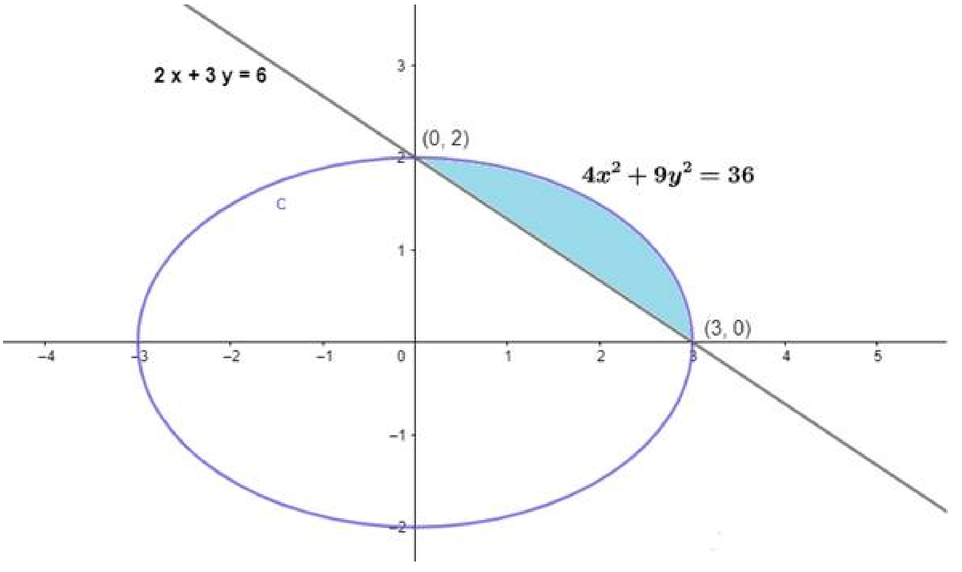
$\frac{x^2}{9} + \frac{y^2}{4} = 1$ (i)

$\frac{x}{3} + \frac{y}{2} = 1$ (ii)

Equation (1) represents an ellipse with centre at origin and meets axes at (±3, 0), (0,±2).

Equation (2) is a line that meets axes at (3, 0), (0, 2).

The sketch of the two curves are shown below:



Required area =

$= \int_0^3 (y_1 - y_2) dx$

$= \int_0^3 \left[\frac{2}{3}\sqrt{0 - x^2} dx - \frac{2}{3}(3 - x) dx\right]$

$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - 3x + \frac{x^2}{2}\right]_0^3$

$= \frac{2}{3}\left[\left\{0 + \frac{9}{2}.\frac{\pi}{2} - 9 + \frac{9}{2}\right\} - \{0\}\right]$

$= \frac{2}{3}\left[\frac{9\pi}{4} - \frac{9}{2}\right]$

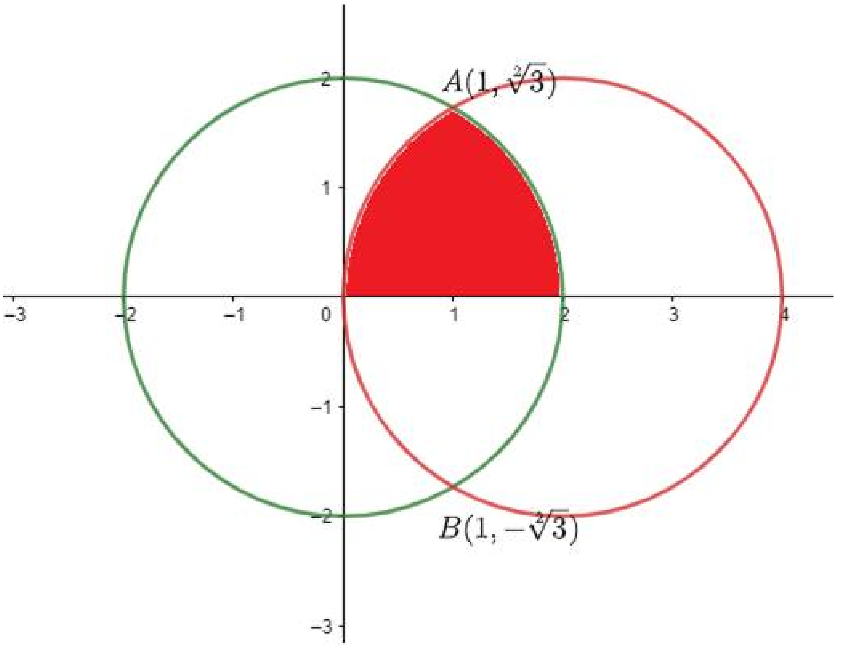
$= \left(\frac{3\pi}{2} - 3\right)$ sq. units

The area of the region: $\left\{(x,y): \frac{x^2}{9} + \frac{y^2}{4} \le 1 \le \frac{x}{3} + \frac{y}{2}\right\}$ is $\left(\frac{3\pi}{2} - 3\right)$ sq. units

45. Question

Using integration find the area of the region bounded by the curve $y = \sqrt{4 - x^2}$, $x^2 + y^2 - 4x = 0$ and the x-axis.

Answer



First, let us find the intersection points of the curve,

Given Equations are $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x = 0$.

From both of the equations,

$4x = 4$

$x = 1$

Putting this value in $x^2 + y^2 = 4$, we get,

$1 + y^2 = 4$

$y^2 = 3$

$y = \pm\sqrt{3}$

Thus the curves intersect at A(1, √3) and B(1, - √3)

The area to be found is shaded in the figure above.

Area of Shaded region = $\int_0^1 \sqrt{4-x^2}\,dx + \int_1^2 \sqrt{4-(x-2)^2}\,dx$

$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 + \left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_1^2$

$= \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{3}$

$= 2\frac{\pi}{3}$

Area of Shaded region = $2\frac{\pi}{3}$ square units.

46. Question

Find the area enclosed by the curves y = |x – 1| and y = – |x – 1| + 1.

Answer

To find the area enclosed by

y = |x – 1|

$$y = \begin{cases} -(x-1), \text{if} x-1 < 0 \\ (x-1), \text{if} x-1 \geq 0 \end{cases}$$

$$y = \begin{cases} 1-x, \text{if} x < 1 \ \ldots(i) \\ x-1, \text{if} x \geq 1 \ \ldots(ii) \end{cases}$$

And y = – |x – 1| + 1

$$y = \begin{cases} +(x-1) + 1, \text{if} x-1 < 0 \\ -(x-1) + 1, \text{if} x-1 \geq 0 \end{cases}$$

$$y = \begin{cases} x, \text{if} x < 1 \ \ldots(iii) \\ -x + 2, \text{if} x \geq 1 \ \ldots(iv) \end{cases}$$

Solving both the equation for x<1

Y = 1 – x and y = x,

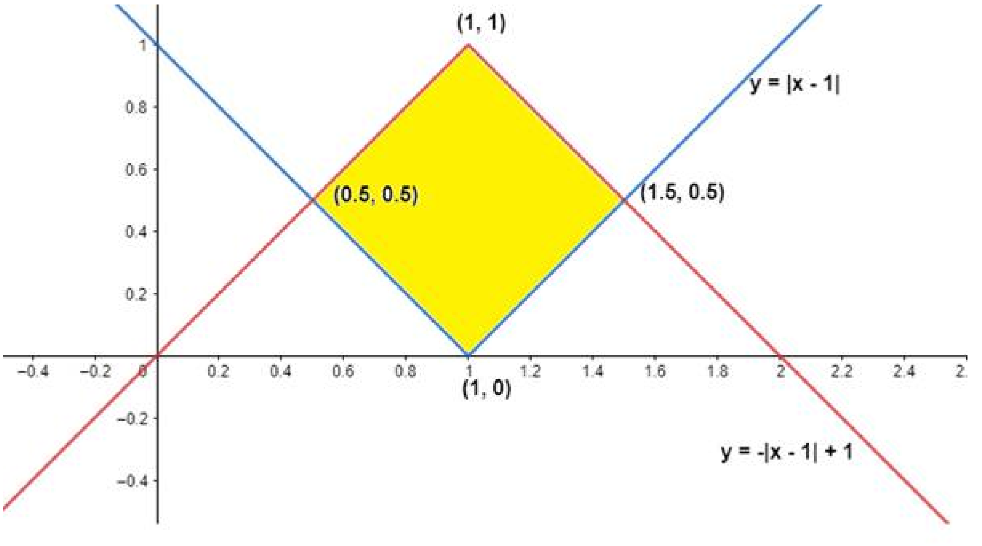
We get $x = \frac{1}{2}$ and $y = \frac{1}{2}$

And solving both the equations for x≥1

Y = x – 1 and y = 2 – x,

We get $x = \frac{3}{2}$ and $y = \frac{1}{2}$

These are shown in the graph below:



Required area = Region ABCDA

Required area = Region BDCB + Region ABDA ...(1)

$$= \int_{\frac{1}{2}}^{1}(y_1 - y_2)dx + \int_{1}^{\frac{3}{2}}(y_3 - y_4)dx$$

$$= \int_{\frac{1}{2}}^{1}(x - 1 + x)dx + \int_{1}^{\frac{3}{2}}(-x + 2 - x + 1)dx$$

$$= [x^2 - x]_{\frac{1}{2}}^{1} + [3x - x^2]_{1}^{\frac{3}{2}}$$

$$= \left[(1-1) - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - (3-1)\right]$$

$$= \frac{1}{4} + \frac{9}{4} - 2$$

$= \frac{1}{2}$ sq. units

The area enclosed by the curves y = |x – 1| and y = – |x – 1| + 1 is $\frac{1}{2}$ sq. units

47. Question

Find the area enclosed by the curves $3x^2 + 5y = 32$ and y = |x – 2|.

Answer

To find area enclosed by

$3x^2 + 5y = 32$

$3x^2 = -5\left(y - \frac{32}{5}\right)$ ...(i)

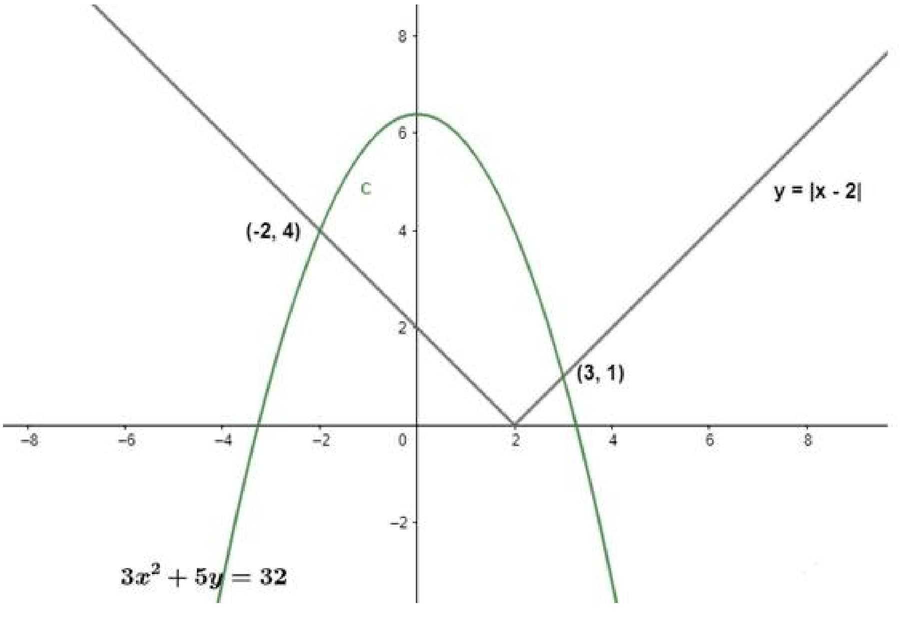
And y = |x – 2|

$$y = \begin{cases} -(x-2), \text{if } x-2 < 1 \\ (x-2), \text{if } x-2 \geq 1 \end{cases}$$

$$y = \begin{cases} 2-x, \text{if } x < 2 \ \ldots(\text{ii}) \\ x-2, \text{if } x \geq 2 \end{cases}$$

Equation (i) represents a parabola with vertex (0, 32/5) and equation (ii) represents lines.

These are shown in the graph below:



Required area = Region ABECDA

= Region ABEA + Region AECDA

$= \int_2^3 (y_3 - y_4)dx + \int_{-2}^{2} (y_1 - y_2)dx$

$= \int_2^3 \left(\frac{32 - 3x^2}{5} - x + 2\right) dx + \int_{-2}^{2} \left(\frac{32 - 3x^2}{5} - 2 + x\right) dx$

$= \int_2^3 \left(\frac{32 - 3x^2 - 5x + 10}{5}\right) dx + \int_{-2}^{2} \left(\frac{32 - 3x^2 - 10 + 5x}{5}\right) dx$

$= \frac{1}{5}\left[\int_2^3 (42 - 3x^2 - 5x)dx + \int_{-2}^{2} (22 - 3x^2 + 5x)dx\right]$

$$= \frac{1}{5}\left[\left(42x - x^3 - \frac{5x^2}{2}\right)_2^3 + \left(22x - x^3 - \frac{5x^2}{2}\right)_{-2}^2\right]$$

$$= \frac{1}{5}\left[\left\{\left(126 - 27 - \frac{45}{2}\right) - (84 - 8 - 10)\right\} + \{(44 - 8 + 10) - (-44 + 8 + 10)\}\right]$$

$$= \frac{1}{5}\left[\left\{\frac{153}{2} - 66\right\} + \{46 + 26\}\right]$$

$$= \frac{1}{5}\left[\frac{21}{2} + 72\right]$$

$$= \frac{33}{2}\text{sq. units}$$

The area enclosed by the curves $3x^2 + 5y = 32$ and $y = |x - 2|$ is $\frac{33}{2}$ sq. units

48. Question

Find the area enclosed by the parabolas $y = 4x - x^2$ and $y = x^2 - x$.

Answer

To area enclosed by

$Y = 4x - x^2$ (1)

$4x - x^2 = x^2 - x$

$2x^2 - 5x = 0$

$x = 0$ or $x = \frac{5}{2}$

$y = 0$ or $y = \frac{15}{4}$

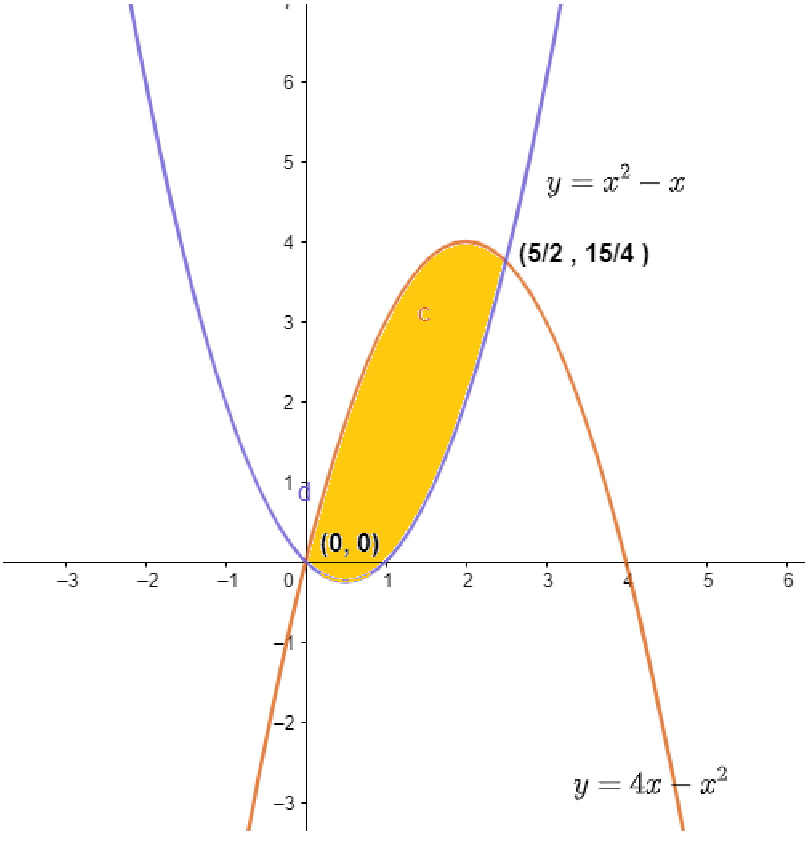
And $y = x^2 - x$

$\left(y + \frac{1}{4}\right) + \left(x - \frac{1}{2}\right)^2$ ...(2)

Equation (1) represents a parabola downward with vertex at (2,4) and meets axes at (4,0), (0,0).

Equation (2) represents a parabola upward whose vertex is $\left(\frac{1}{2}, \frac{1}{4}\right)$ and meets axes at Q(1,0), (0,0).Points of intersection of parabolas are O (0,0) and A $\left(\frac{5}{2}, \frac{15}{4}\right)$.

These are shown in the graph below:



Required area = Region OQAP

$$A = \int_0^{\frac{5}{2}} (y_1 - y_2)dx$$

$$= \int_0^{\frac{5}{2}} [4x - x^2 - x^2 + x]dx$$

$$= \int_0^{\frac{5}{2}} [5x - 2x^2]dx$$

$$= \left[\frac{5x^2}{2} - \frac{2}{3}x^3\right]_0^{\frac{5}{2}}$$

$$= \left[\left(\frac{125}{8} - \frac{250}{24}\right) - (0)\right]$$

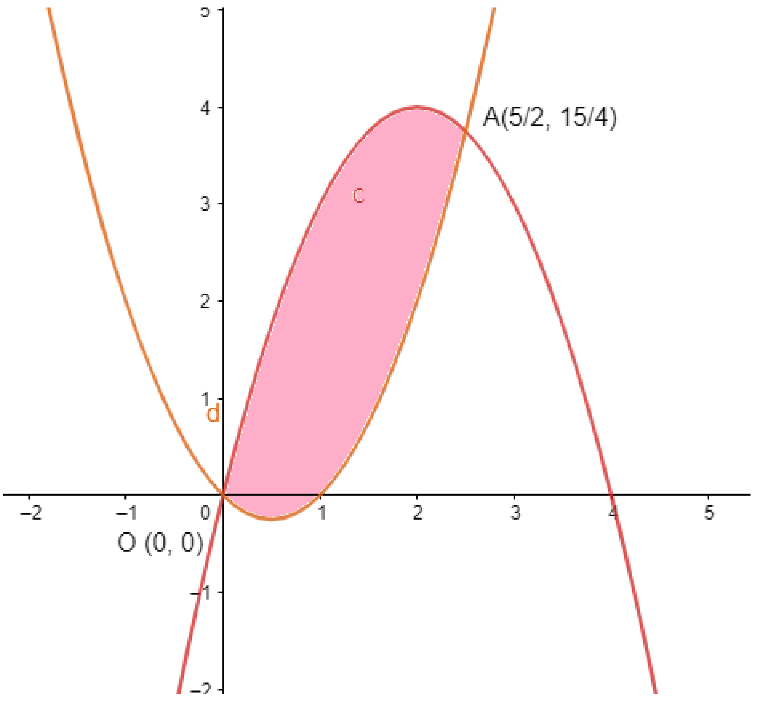
$$A = \frac{125}{24} \text{sq. units}$$

The area enclosed by the parabolas y = 4x – x^2 and y = x^2 – x is $\frac{125}{24}$ sq. units.

49. Question

In what ratio does the x-axis divide the area of the region bounded by the parabolas y = 4x – x^2 and y = x^2 – x?

Answer



Let us find the intersection points first,

We have the equations of curves,

$y = 4x - x^2$ ....(1)

$y = x^2 - x$ ....(2)

From (1) and (2) we can get,

$x^2 - x = 4x - x^2$

$2x^2 - 5x = 0$

$x(2x - 5) = 0$

$x = 0$ or $x = 5/2$

Putting these values of x in equation (2) we get,

At $x = 0$,

$Y = 0^2 - 0 = 0$

At $x = 5/2$

$$y = \frac{25}{4} - \frac{5}{2} = \frac{15}{4}$$

Hence intersection points are (0, 0) and $\left(\frac{5}{2}, \frac{15}{4}\right)$

Area bounded by the curves is shown by the shaded region of the figure shown above.

Area of shaded region = $\int_0^{\frac{5}{2}} \{[4x - x^2] - [x^2 - x]\}\, dx$

Area of shaded region = $\int_0^{\frac{5}{2}} (5x - 2x^2)dx$

Area of Shaded Region = $\left[\frac{5x^2}{2} - \frac{2}{3}x^3\right]_0^{\frac{5}{2}}$

Area of Shaded Region = $\frac{125}{4}$ Square units.

50. Question

Find the area of the figure bounded by the curves y = |x – 1| and y = 3 – |x|.

Answer

To find the area of the figure bounded by

y = |x – 1|

$$\Rightarrow y = \begin{cases} (x-1), \text{if} x \geq 1 \\ -(x-1), \text{if} x<1 \end{cases}$$

Y = x – 1 is a straight line passing through A(1,0)

Y = 1 – x is a straight line passing through A(1,0) and cutting y - axis at B(0,1)

y = 3 – |x|

$$\Rightarrow y = \begin{cases} 3-x, \text{if} x \geq 1 \\ 3-(-x) = 3 + x, \text{if} x<1 \end{cases}$$

Y = 3 – x is a straight line passing through C(0,3)and O(3,0)

Y = 3 + x is a straight line passing through C(0,3)and D(– 3,0)

Point of intersection for

Y = x – 1

And y = 3 – x

We get

X – 1 = 3 – x

or, 2x – 4 = 0

or, x = 2

or, y = 2 – 1 = 1

Thus, point of intersection for y = x – 1 and y = 3 + x is B(2,1)

Point of intersection for

y = 1 – x

y = 3 + x

or,1 – x = 3 + x

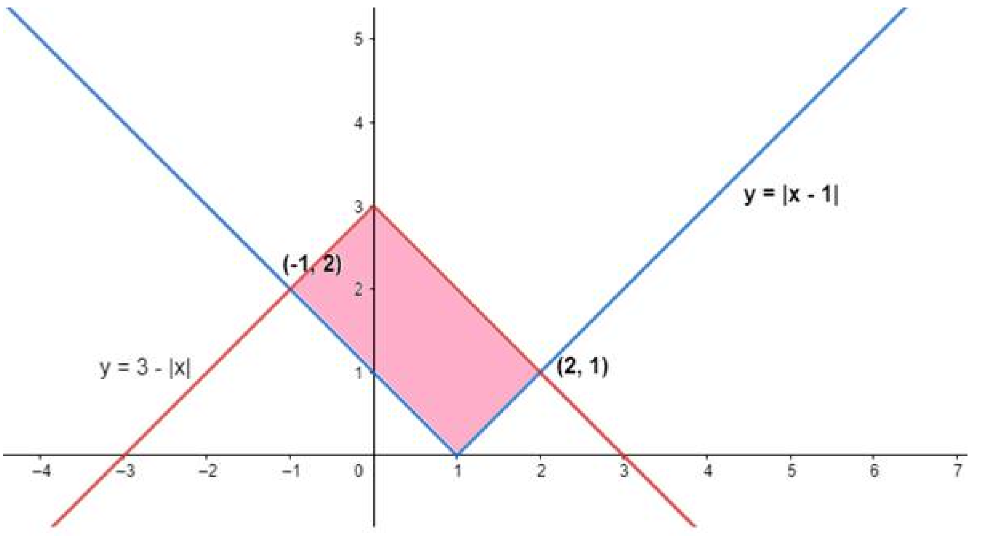
or, 2x = – 2

or, x = – 1

or, y = 1 – (– 1) = 2

Thus, point of intersection for y = 1 – x and y = 3 + x is D(– 1,2)

These are shown in the graph below:



Required area = Region ABCDA

= Region ABFA + Region AFCEA + Region CDEC

$= \int_1^2 (y_1 - y_2)dx + \int_0^1 (y_1 - y_3)dx + \int_{-1}^0 (y_4 - y_3)dx$

$= \int_1^2 (3 - x - x + 1)dx + \int_0^1 (3 - x - 1 + x)dx + \int_{-1}^0 (3 + x - 1 + x)dx$

$= \int_1^2 (4 - 2x)dx + \int_0^1 2dx + \int_{-1}^0 (2 + 2x)dx$

$= [4x - x^2]_1^2 [2x]_0^1 + [2x + x^2]_{-1}^0$

$= [(8 - 4) - (4 - 1)] + [2 - 0] + [(0) - (-2 + 1)]$

$= (4 - 3) + 2 + 1$

$A = 4 \text{ sq. unit}$

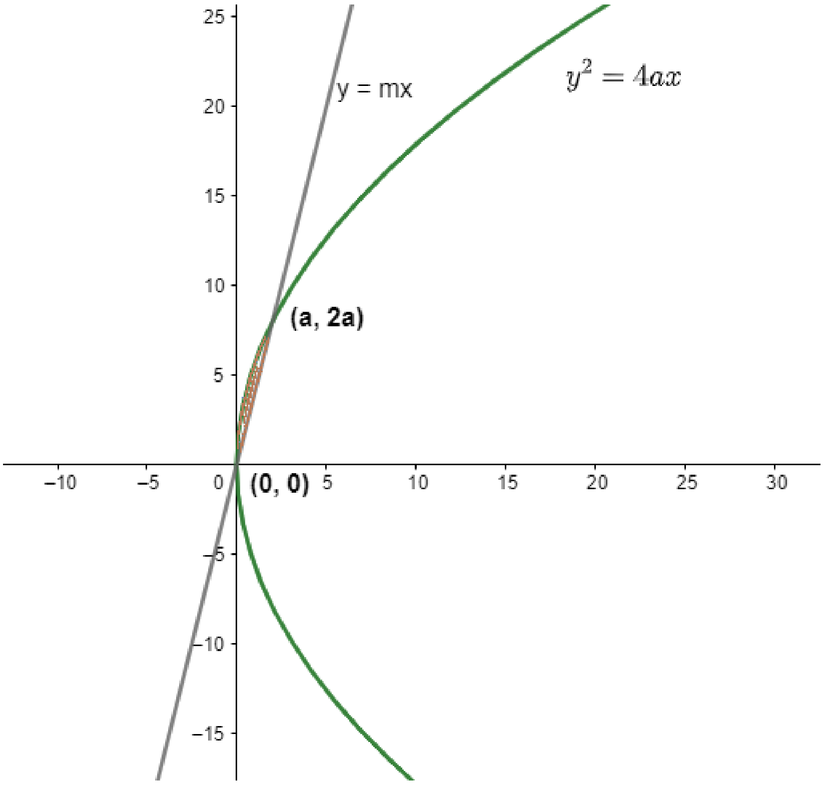
The area of the figure bounded by the curves y = |x – 1| and y = 3 – |x| is 4 sq. units

51. Question

If the area bounded by the parabola $y^2 = 4ax$ and the line y = mx is $a^2/12$ sq. Units, then using integration, find the value of m.

Answer

Area of the bounded region = $a^2/12$



Mathematically the area in integral form will be, $\int_0^a \sqrt{4ax} - mxdx$,

So, on equating..

$$\frac{a^2}{12} = \int_0^a \sqrt{4ax} - mxdx$$

$$\frac{a^2}{12} = \left[2\sqrt{a}\frac{x^{1/2}}{3/2} - m\frac{x^2}{2}\right]_0^a$$

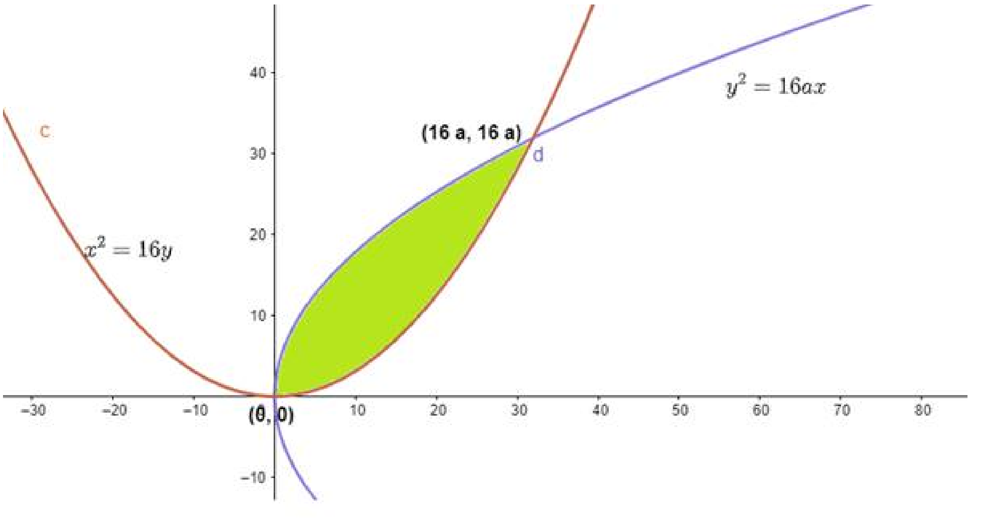
$$\frac{a^2}{12} = \frac{4a^2}{3} - m\frac{a^2}{2}$$

$$m = 2$$

52. Question

If the area enclosed by the parabolas $y^2 = 16ax$ and $x^2 = 16ay$, a>0 is 1024/3 square units, find the value of a.

Answer



Area of the bounded region - 1024/3

$$\frac{1024}{3} = \int_0^{16a} \sqrt{16ax} - \frac{x^2}{16a} dx$$

$$\frac{1024}{3} = \left[4\sqrt{a}\frac{x^{1/2}}{3/2} - \frac{x^3}{48a}\right]_0^{16a}$$

$$\frac{1024}{3} = \frac{(16a)^2 x2}{3} - \frac{(16a)^3}{48a}$$

$a = 2$

Exercise 21.4

1. Question

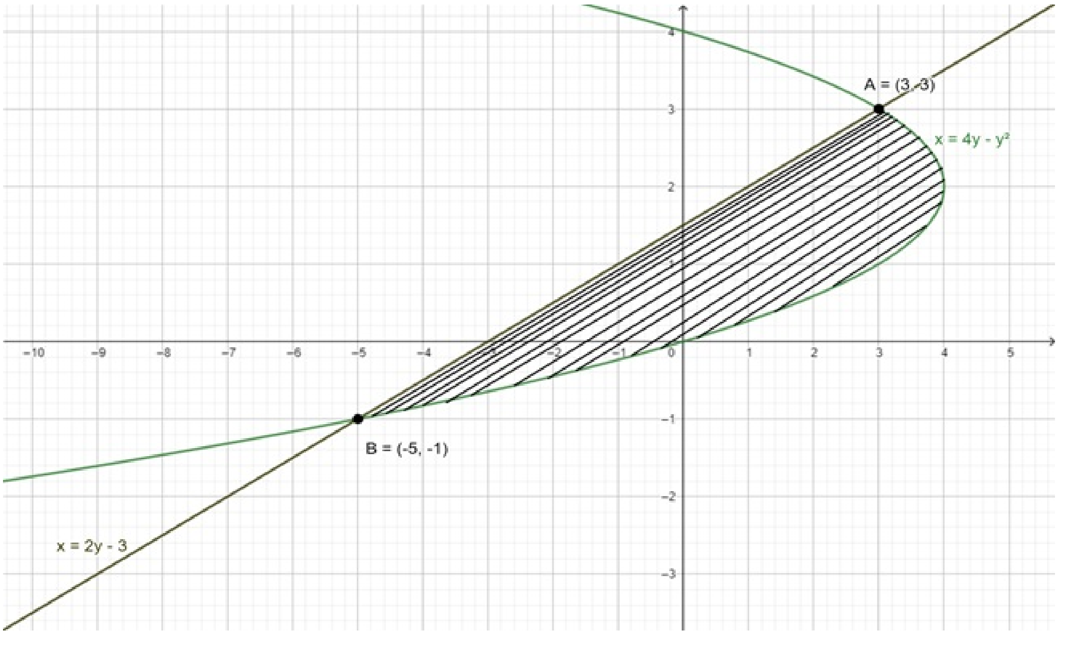
Find the area of the region between the parabola $x = 4y - y^2$ and the line $x = 2y - 3$.

Answer

Given: - Two equation; Parabola $x = 4y - y^2$ and Line $x = 2y - 3$

Now to find an area between these two curves, we have to find a common area or the shaded part.

From figure, we can see that,



Area of shaded portion = Area under the parabolic curve – Area under line

Now, Intersection points;

From parabola and line equation equate x, we get

$\Rightarrow 4y - y^2 = 2y - 3$

$\Rightarrow y^2 - 2y - 3 = 0$

$\Rightarrow y^2 - 3y + y - 3 = 0$

$\Rightarrow y(y - 3) + 1(y - 3)$

$\Rightarrow (y + 1)(y - 3)$

$\Rightarrow y = -1, 3$

So, by putting the value of x in any curve equation, we get,

$\Rightarrow x = 2y - 3$

For y = – 1

⇒ x = 2(– 1) – 3

⇒ x = – 5

For

y = 3

⇒ x = 2(3) – 3

⇒ x = 3

Therefore two intersection points coordinates are (– 5, – 1) and (3, 3)

Area of the bounded region

= (Area under the parabola curve from – 1 to 3) – (Area under line from – 1 to 3)

Tip: - Take limits as per strips. If strip is horizontal than take y limits or if integrating with respect to y then limits are of y.

Here, limits are for y i.e from - 1 to 3.

$$= \int_{-1}^{3} x\,dy \text{ \{i.e curve under parabola\}} - \int_{-1}^{3} x\,dy \text{ \{i. e curve under line\}}$$

$$= \int_{-1}^{3}(4y - y^2)dy - \int_{-1}^{3}(2y - 3)dy$$

$$= \int_{-1}^{3}(4y)dy - \int_{-1}^{3}(y^2)dy - \int_{-1}^{3}(2y)dy + \int_{-1}^{3}(3)dy$$

$$= \frac{4}{2}[y^2]_{-1}^{3} - \frac{1}{3}[y^3]_{-1}^{3} - \frac{2}{2}[y^2]_{-1}^{3} + 3[y]_{-1}^{3}$$

$$= 2[y^2]_{-1}^{3} - \frac{1}{3}[y^3]_{-1}^{3} - [y^2]_{-1}^{3} + 3[y]_{-1}^{3}$$

Now putting limits, we get

$$= 2(32 - 1) - \frac{1}{3}\{33 - (-1)3\} - (32 - 1) + 3\{3 - (-1)\}$$

$$= 2(9 - 1) - \frac{1}{3}(27 + 1) - (9 - 1) + 3(3 + 1)$$

$$= 2(8) - \frac{1}{3}(28) - 8 + 3(4)$$

$$= 8 + 12 - \frac{1}{3}28$$

$$= \frac{60 - 28}{3}$$

$$= \frac{32}{3} \text{ sq units}$$

2. Question

Find the area bounded by the parabola $x = 8 + 2y - y^2$; the y - axis and the lines y = – 1 and y = 3.

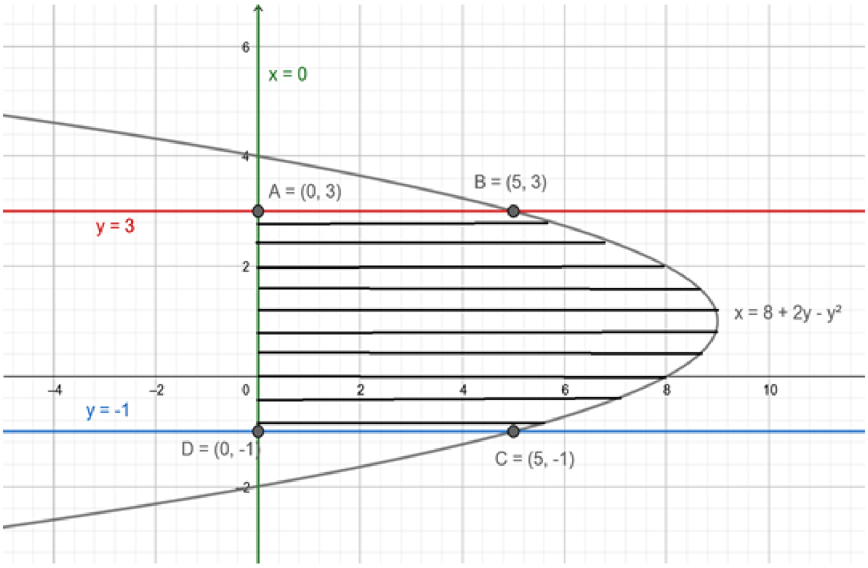
Answer

Given: - Two equation;

Parabola $x = 8 + 2y - y^2$,

y - axis,

Line1 y = – 1, and Line2 y = 3



Now to find the area between these four curves, we have to find a common area (ABDC) or the shaded part.

The 1st intersection of a parabola with line y = – 1, we get,

Putting the value of y = 1 in parabolic equation

$\Rightarrow x = 8 + 2y - y^2$

$\Rightarrow x = 8 + 2(-1) - 1$

$\Rightarrow x = 5$

Hence intersection point is **D(5, – 1)**

The 2nd intersection of parabola with y = 3

Putting the value of y in parabola equation

$\Rightarrow x = 8 + 2y - y^2$

$\Rightarrow x = 8 + 2(3) - 3^2$

$\Rightarrow x = 8 + 6 - 9$

$\Rightarrow x = 5$

Hence, intersection point is **C(5,3)**

and other points are A(0,3), B(0, – 1)

From the figure, we can see that, By taking a **horizontal** strip

The area under shaded portion = Area under parabola from y =– 1 to y = 3.

Tip: - Take limits as per strips. If strip is horizontal than take y limits or if integrating concerning y then limits are of y.

Here, limits are for y i.e. from – 1 to 3

$$= \int_{-1}^{3} x\,dy \text{ \{i.e curve under parabola\}}$$

$$= \int_{-1}^{3} (8 + 2y - y^2)dy$$

$$= \int_{-1}^{3}(8)dy + \int_{-1}^{3}(2y)dy - \int_{-1}^{3}(y^2)dy$$

$$= 8[y]_{-1}^{3} + \frac{2}{2}[y^2]_{-1}^{3} - \frac{1}{3}[y^3]_{-1}^{3}$$

$$= 8(3 + 1) + [y^2]_{-1}^{3} - \frac{1}{3}[y^3]_{-1}^{3}$$

Now putting limits, we get,

$$= 8(4) + (32 - 1) - \frac{1}{3}\{33 - (-1)3\}$$

$$= 32 + (9 - 1) - \frac{1}{3}(27 + 1)$$

$$= 32 - \frac{1}{3}(28) + 8$$

$$= 40 - \frac{1}{3}28$$

$$= \frac{120 - 28}{3}$$

$$= \frac{92}{3} \text{ sq units}$$

3. Question

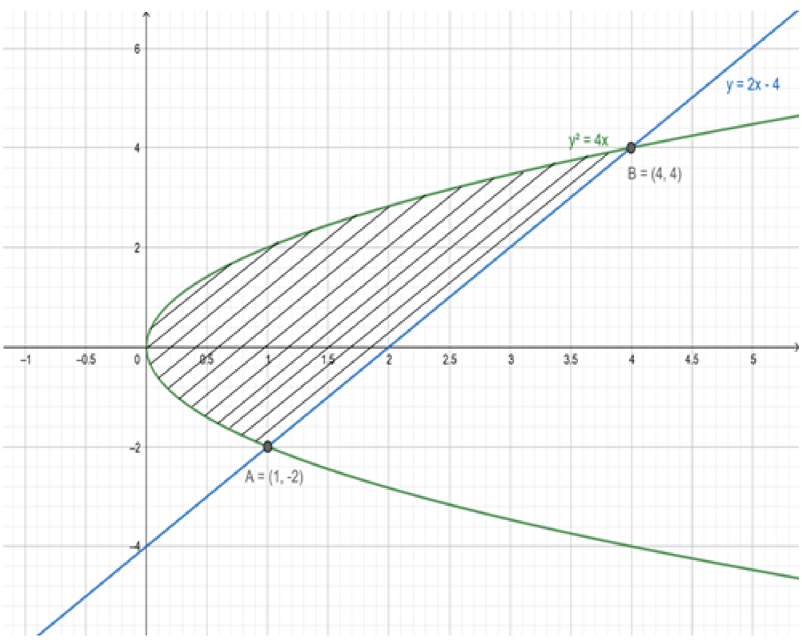
Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

(i) By using horizontal strips

(ii) By using vertical strips.

Answer

Given: - Two curves are $y^2 = 4x$ and $y = 2x - 4$



Now to find the area between these two curves, we have to find common area i.e. Shaded portion

Intersection of parabola $y^2 = 4x$ with line $y = 2x - 4$

Putting the value of y from the equation of a line in parabola equation, we get,

$y^2 = 4x$

$\Rightarrow (2x - 4)^2 = 4x$

$\Rightarrow 4x^2 - 16x + 16 = 4x$

$\Rightarrow 4x^2 - 20x + 16 = 0$

$\Rightarrow 4x^2 - 16x - 4x + 16 = 0$

$\Rightarrow 4x(x - 4) - 4(x - 4) = 0$

$\Rightarrow 4(x - 1)(x - 4) = 0$

$\Rightarrow (x - 1)(x - 4) = 0$

$\Rightarrow x = 1,4$

When $x = 1$, $y = \sqrt{4x}$

$\Rightarrow y = +2, -2$; we take -2 as the intersection is in the 4^{th} quadrant and when $x = 4$, $y = \sqrt{4x}$

$\Rightarrow y = +4, -4$; **we take + 4 as the intersection is in 1 st quadrant**

Therefore intersection points are **B(4,4)** and **C(1, – 2)**

Area of the bounded region, taking strips

i) By using horizontal strips

Therefore, limits are for y and integrating with respect to y

Area bounded by region = {Area under line from – 2 to 4} –{Area under parabola from – 2 to 4}

$$= \int_{-2}^{4} x\,dy\{\text{i.e curve under line}\} - \int_{-2}^{4} x\,dy\,\{\text{i.e curve under parabola}\}$$

$$= \int_{-2}^{4} \frac{y+4}{2}dy - \int_{-2}^{4} \frac{y^2}{4}dy$$

$$= \frac{1}{2}\int_{-2}^{4}(y)dy + \frac{1}{2}\int_{-2}^{4} 4\,dy - \frac{1}{4}\int_{-2}^{4}(y^2)dy$$

$$= \frac{1}{2\times 2}[y^2]_{-2}^{4} + 2[y]_{-2}^{4} - \frac{1}{4\times 3}[y^3]_{-2}^{4}$$

Putting limits, we get

$$= \frac{1}{4}(42 - 4) + 2\{4 - (-2)\} - \frac{1}{12}\{43 - (-2)3\}$$

$$= \frac{1}{4}(16 - 4) + 2(4 + 2) - \frac{1}{12}(64 + 8)$$

$$= \frac{1}{4}(12) + 2(6) - \frac{1}{12}(72)$$

$$= 3 + 12 - 6$$

$$= \mathbf{9\ sq\ units}$$

ii) By using vertical strips.

Therefore, limits are for x, and integrating with respect to x

Area bounded by region = **{**2(Area under parabola from 0 to 1) + (Area under parabola from 1 to 4)**} – {**Area under line from 1 to 4**}**

Tip: - Parabola is symmetrical about x - axis therefore its area is twice the area above x - axis. So, till its latus rectum i.e here a = 1, area is twice the area above x - axis.

$$= 2\int_0^1 y\,dx \text{ \{i.e curve under Parabola\}} + \int_1^4 y\,dx \text{ \{i.e curve under parabola\}}$$
$$- \int_1^4 y\,dx \text{ \{i.e curve under Line\}}$$

$$= 2\int_0^1 2(\sqrt{x})dy + \int_1^4 2(\sqrt{x})dy - \int_1^4 (2x-4)dy$$

$$= 4\int_0^1 (\sqrt{x})dy + 2\int_1^4 (\sqrt{x})dy - 2\int_1^4 (x)dy + \int_1^4 (4)dy$$

$$= \frac{4\times 2}{3}[(\sqrt{x})^2]_0^1 + \frac{4}{3}[\sqrt{x}]_1^4 - \frac{2}{2}[x^2]_1^4 + 4[x]_1^4$$

Putting limits, we get,

$$= \frac{8}{3}(1-0) + \frac{4}{3}\{(\sqrt{4})3 - 1\} - (42-1) + 4(4-1)$$

$$= \frac{8}{3} + \frac{4}{3}(8-1) - (16-1) + 4(3)$$

$$= \frac{8}{3} + \frac{4\times 7}{3} - 15 + 12$$

$$= \frac{8}{3} + \frac{28}{3} - 3$$

$$= \frac{36-9}{3}$$

$$= \frac{27}{3}$$

$$= 9 \text{ sq units}$$

Hence from both methods we get same answer

4. Question

Find the area of the region bounded by the parabola $y^2 = 2x$ and the straight-line x – y = 4.

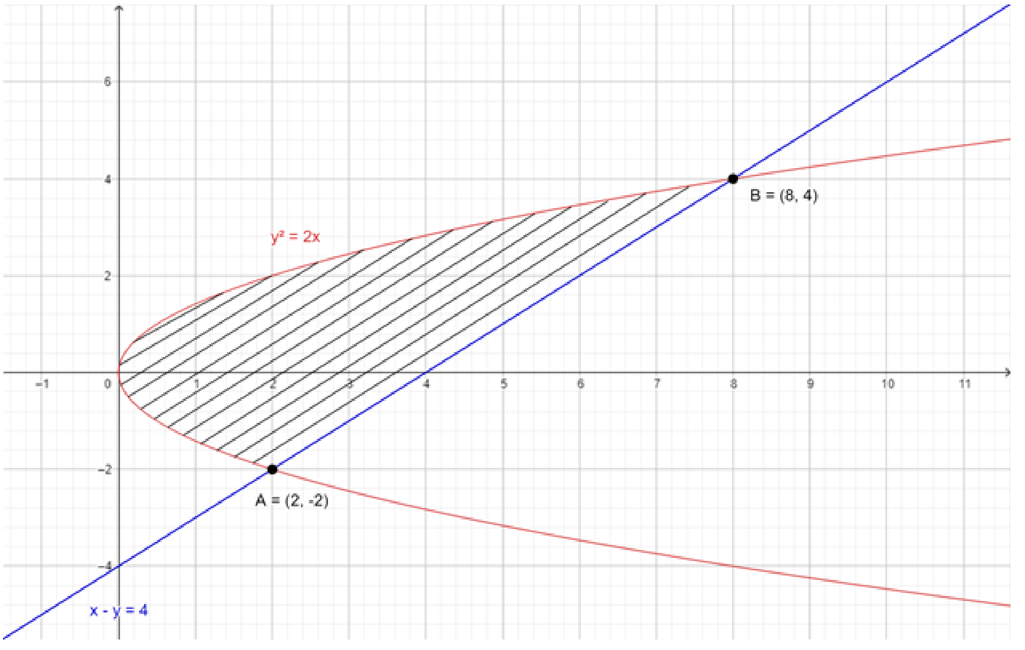
Answer

Given: -

Two equation;

Parabola $y^2 = 2x$ and

Line x – y = 4



Now to find an area between these two curves, we have to find a common area or the shaded part.

From figure we can see that,

Area of shaded portion = Area under line curve – Area under parabola; horizontal strip

Now, Intersection points;

From parabola and line equation equate y, x – 4 = y we get

$\Rightarrow y^2 = 2x$

$\Rightarrow (x - 4)^2 = 2x$

$\Rightarrow x^2 - 8x + 16 = 2x$

$\Rightarrow x^2 - 10x + 16 = 0$

$\Rightarrow x^2 - 8x - 2x + 16 = 0$

$\Rightarrow x(x - 8) - 2(x - 8) = 0$

$\Rightarrow (x - 8)(x - 2) = 0$

$\Rightarrow x = 8,2$

So, by putting the value of x in any curve equation, we get,

$\Rightarrow y = x - 4$

For x = 8

$\Rightarrow y = 8 - 4$

$\Rightarrow y = 4$

For x = 2

$\Rightarrow y = 2 - 4$

$\Rightarrow y = -2$

Therefore, two intersection points coordinates are (8, 4) and (2, – 2)

Area of the bounded region

= Area under the line curve from – 2 to 4 – Area under parabola from – 2 to 4

Tip: - Take limits as per strips. If the strip is horizontal than take y limits or if integrating with respect to y then limits are of y.

Area bounded by region = {Area under line from – 2 to 4} – {Area under parabola from – 2 to 4}

$= \int_{-2}^{4} x\,dy\ \{\text{i.e curve under line}\} - \int_{-2}^{4} x\,dy\ \{\text{i.e curve under parabola}\}$

$= \int_{-2}^{4} (y + 4)dy - \int_{-2}^{4} \frac{y^2}{2} dy$

$= \int_{-2}^{4} (y)dy + \int_{-2}^{4} 4\,dy - \frac{1}{2}\int_{-2}^{4} (y^2)dy$

$= \frac{1}{2}[y^2]_{-2}^{4} + 4[y]_{-2}^{4} - \frac{1}{2 \times 3}[y^3]_{-2}^{4}$

Putting limits, we get

$= \frac{1}{2}(42 - 4) + 4\{4 - (-2)\} - \frac{1}{6}\{43 - (-2)3\}$

$= \frac{1}{2}(16 - 4) + 4(4 + 2) - \frac{1}{6}(64 + 8)$

$= \frac{1}{2}(12) + 4(6) - \frac{1}{6}(72)$

= 6 + 24 – 12

= 18 sq units

MCQ

1. Question

If the area above the x-axis, bounded by the curves $y = 2^{kx}$ and x = 0, and x = 2 is $\frac{3}{\log_e 2}$, then the value of k is

A. 1/2

B. 1

C. -1

D. 2

Answer

The area can be computed as –

$A = \int_{x=0}^{x=2} y\,dx$

$= \int_{x=0}^{x=2} 2^{kx}\,dx$

$= \left[\frac{1}{k} \cdot \frac{2^{kx}}{\log_e 2}\right]_{x=0}^{x=2}$

$= \frac{1}{k} \cdot \frac{2^{2k} - 1}{\log_e 2}$

Comparing with $\frac{3}{\log_e 2}$

$\frac{2^{2k} - 1}{k} = 3$

$\Rightarrow 4^k = 3^k + 1$

Using Binomial Theorem,

$3^k + 1 = 4^k = (1+3)^k$

$= 1 + k \cdot 3 + \frac{k(k-1)}{2} \cdot 3^2 + \cdots + \frac{k(k-1)}{2} \cdot 3^{k-2} + k \cdot 3^{k-1} + 3^k$

This equality holds only when k = 1, because only then you get two terms in the expansion.

Ans: k = 1

2. Question

The area included between the parabolas $y^2 = 4x$ and $x^2 = 4y$ is (in square units)

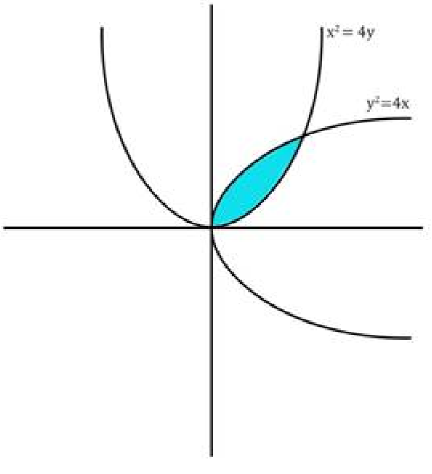
A. 4/3

B. 1/3

C. 16/3

D. 8/3

Answer



The blue area is what we need to compute. To do that we need the bounds, i.e., where the area starts and where it ends. At the points of intersection, both the equations are satisfied.

This means that at points of intersection

$y^2 = 4x$

$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x$ (from the other equation)

Let us solve this.

$\left(\frac{x^2}{4}\right)^2 = 4x$

$x^4 = 64x$

$\Rightarrow x(x^3 - 64) = 0$

$\Rightarrow x = 0$ or $x^3 = 64$, i.e, $x = 4$

So the points of intersection are (0,0) and (4,4)

Now, let's compute the area.

If we integrate w.r.t x, we'll have to integrate the space between the two curves from x = 0 to x =4.

i.e.,

$$A=\int_{x=0}^{x=4}\left(2\sqrt{x}-\frac{x^2}{4}\right)dx$$

$$=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{x^3}{12}\right]_{x=0}^{x=4}$$

$$=\left[\frac{4}{3}\cdot\left(4^{\frac{3}{2}}-0^{\frac{3}{2}}\right)-\left(\frac{4^3}{12}-\frac{0^3}{12}\right)\right]$$

$$=\left[\frac{4}{3}\cdot 8-\frac{16}{3}\right]$$

$$=\left[\frac{32}{3}-\frac{16}{3}\right]$$

$=\frac{16}{3}$ (Ans)

3. Question

The area bounded by the curve y = $\log_e$ x and x-axis and the straight line x = e is

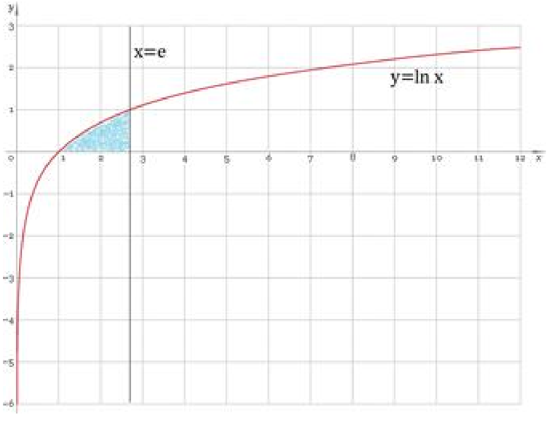
A. e sq. units

B. 1 sq. units

C. $1-\frac{1}{e}$ sq. units

D. $1+\frac{1}{e}$ sq. units

Answer



We need to find the area of the blue shaded region.

At, x = 1, y = $\log_e(1)$ = 0

And, at x = e, y = $\log_e(e)$ = 1

These are our bounds.

So, this will be computed as –

$$A=\int_{x=1}^{x=e}\log_e x\,dx$$

Using Integration by parts,

$$A=\left[\log_e x\int dx-\int\frac{1}{x}\cdot x\,dx\right]_{x=1}^{x=e}$$

$= [x\log_e x - x]_{x=1}^{x=e}$

=[e – e – 0 + 1]

= 1 (Ans)

4. Question

The area bounded by y = 2 – x^2 and x + y = 0 is

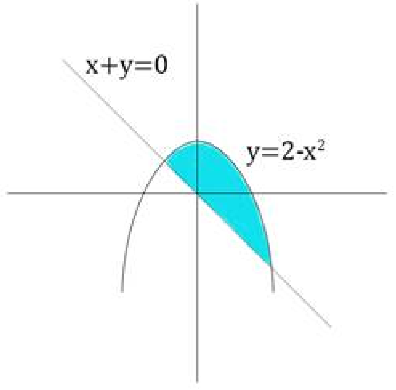
A. $\frac{7}{2}$ sq. units

B. $\frac{9}{2}$ sq. units

C. 9 sq. units

D. none of these

Answer



- the blue shaded region above

To define the bounds, we need to find the points of intersection. We know that at the points of intersection, both the equations are satisfied.

⇒ x + y = 0

⇒ x + (2 – x^2) = 0 (from the other equation)

So, x^2 – x – 2 = 0 i.e., (x - 2)(x + 1) = 0 or x = -1,2

So, bounds are x = -1 to x = 2

Therefore, area shall be evaluated as –

$$\int_{x=-1}^{x=2} \{(2-x^2)-(-x)\}\,dx = \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{x=-1}^{x=2}$$

$$= \left[2\cdot 2 - \frac{2^3}{3} + \frac{2^2}{2} - 2\cdot -1 + \frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right]$$

$= 6 - \frac{9}{3} + \frac{3}{2} = \frac{9}{2}\ sq.units$ (Ans)

5. Question

The area bounded by the parabola x = 4 – y^2 and y-axis, in square units, is

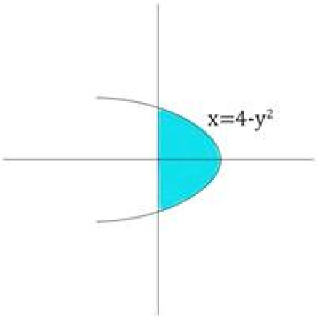
A. 3/32

B. 32/3

C. 33/2

D. 16/3

Answer



Is the blue shaded region.

At points of intersection on y-axis, x = 0

So, equation becomes 0 = 4 – y^2 or y = -2, 2

This becomes our bounds, and we integrate w.r.t. the y-axis –

$$A = \int_{y=-2}^{y=2} x\,dy = \int_{y=-2}^{y=2} (4 - y^2)\,dy = \left[4y - \frac{y^3}{3}\right]_{y=-2}^{y=2}$$

$$= \left[4 \cdot 2 - \frac{2^3}{3} - 4 \cdot -2 + \frac{(-2)^3}{3}\right]$$

$$= 16 - \frac{16}{3}$$

$= \frac{32}{3}\ sq.units$ (Ans)

6. Question

If A_n be the area bounded by the curve y = $(\tan x)^n$ and the lines x = 0, y = 0 and x = π/4, then for x > 2

A. $A_n + A_{n-2} = \frac{1}{n-1}$

B. $A_n + A_{n-2} < \frac{1}{n-1}$

C. $A_n - A_{n-2} = \frac{1}{n-1}$

D. none of these

Answer

$$A_n = \int_{x=0}^{x=\frac{\pi}{4}} (\tan x)^n\,dx = \int_{x=0}^{x=\frac{\pi}{4}} (\tan x)^{n-2}(\tan^2 x)\,dx = \int_{x=0}^{x=\frac{\pi}{4}} (\tan x)^{n-2}(\sec^2 x - 1)\,dx$$

$$= \int_{x=0}^{x=\frac{\pi}{4}} (\tan x)^{n-2}(\sec^2 x)\,dx - \int_{x=0}^{x=\frac{\pi}{4}} (\tan x)^{n-2}\,dx = \int_{x=0}^{x=\frac{\pi}{4}} (\tan x)^{n-2}(\sec^2 x)\,dx - A_{n-2}$$

$$\Rightarrow A_n + A_{n-2} = \int_{x=0}^{x=\frac{\pi}{4}} (\tan x)^{n-2}(\sec^2 x)\,dx$$

Now, let u = tan x ⇒ du = $\sec^2$ x dx

When x = 0, u = 0 and when x= π/4, u = 1

So, $A_n + A_{n-2} = \int_{u=0}^{u=1} u^{n-2}\, du$

$= \left[\frac{u^{n-1}}{n-1}\right]_{u=0}^{u=1}$

$= \left[\frac{1^{n-1}}{n-1} - \frac{0^{n-1}}{n-1}\right]$

$= \frac{1}{n-1}$

Ans: $A_n \; + A_n - 2 \; = \frac{1}{n-1}$

7. Question

The area of the region formed by $x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$ and $x \le 5/2$ is

A. $\frac{\pi}{6} - \frac{\sqrt{3}+1}{8}$

B. $\frac{\pi}{6} + \frac{\sqrt{3}+1}{8}$

C. $\frac{\pi}{6} - \frac{\sqrt{3}-1}{8}$

D. none of these

Answer

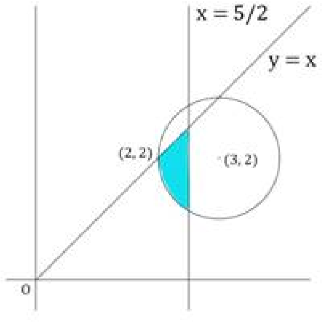
$x^2 + y^2 - 6x - 4y + 12 \le 0$ can be written as –

$x^2 - 6x + 9 - 9 + y^2 - 4y + 4 - 4 + 12 \le 0$

i.e., $(x - 3)^2 + (y - 2)^2 \le 1$

So, this indicates the area enclosed by a circle centred at (3,2) with radius 1.

Now, the area of the region formed by $x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$ and $x \le 5/2$ is



So, our bounds are x = 2 to x = 2.5

The equation for the ordinate of point on the circle from x = 2 to x = 2.5 –

$(x - 3)^2 + (y - 2)^2 = 1$

$\Rightarrow (y - 2)^2 = 1 - (x - 3)^2$

$\Rightarrow y - 2 = \pm\sqrt{1-(x-3)^2} \Rightarrow y = 2 \pm\sqrt{1-(x-3)^2}$

Since we are considering the lower value of y in x = 2 to x = 2.5 (the one iny ≤ x),

$y = 2 - \sqrt{1-(x-3)^2}$

So, the area is –

$$\int_{x=2}^{x=2.5}\left\{(x)-\left(2-\sqrt{1-(x-3)^2}\right)\right\}dx = \int_{x=2}^{x=2.5}\left\{x-2+\sqrt{1-(x-3)^2}\right\}dx$$

$$=\left[\frac{x^2}{2}-2x+\frac{x-3}{2}\sqrt{1-(x-3)^2}+\frac{1}{2}\sin^{-1}(x-3)\right]_{x=2}^{x=2.5}$$

$$=\left[\frac{6.25}{2}-5-\frac{1}{4}\sqrt{1-0.25}+\frac{1}{2}\sin^{-1}(-0.5)-\frac{4}{2}+4+\frac{1}{2}\sqrt{1-1}-\frac{1}{2}\sin^{-1}(-1)\right]$$

$=\frac{2.25}{2}-1-\frac{\sqrt{3}}{8}-\frac{\pi}{12}+\frac{\pi}{4}=\frac{\pi}{6}-\frac{\sqrt{3}-1}{8}$ (Ans)

8. Question

The area enclosed between the curves $y = \log_e (x + e)$, $x = \log_e\left(\frac{1}{y}\right)$ and the x-axis is

A. 2

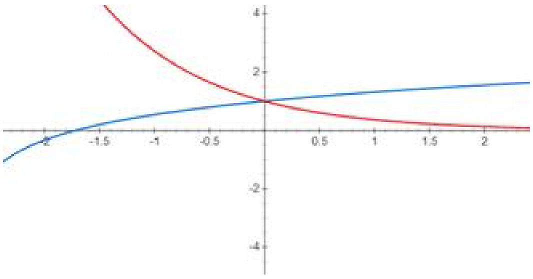
B. 1

C. 4

D. none of these

Answer

$y = \log_e (x + e)$ and $x = \log_e\left(\frac{1}{y}\right)$ look like –



The curves intersect at (0, 1)

(Putting x = 0 in the 2 curves, $y = \log_e(e) = 1$ and $0 = \log_e(1/y)$,

i.e., $y = 1/e^0 = 1$)

So, bounds are x = 1 – e to x = 0 for the first curve and then x = 0 to apparently x = ∞ for the second curve.

Therefore,

$$A=\int_{x=1-e}^{x=0}\log_e(x+e)\,dx+\lim_{B\to\infty}\int_{x=0}^{x=B}e^{-x}\,dx$$

$$=[(x+e)\{\log_e(x+e)-1\}]_{x=1-e}^{x=0}+\lim_{B\to\infty}[-e^{-x}]_{x=0}^{x=B}$$

$=[e\{1-1\}-\{-1\}]+\lim_{B\to\infty}[1-e^{-B}]=[1]+[1-0]=2$ (Ans)

1. Question

The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to it at the point with the ordinate 3 and the x-axis is

A. 3

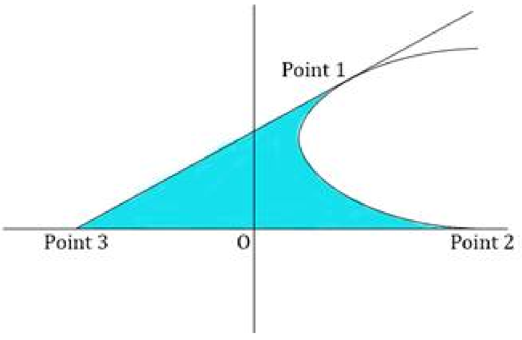
B. 6

C. 7

D. none of these

Answer

The question describes something like –



To solve the problem, we need to find the points 1, 2, and 3 first. They are the key to setting up the bounds of our integration and understanding what function to integrate.

For point 1:-

We know it is a point on the parabola with ordinate 3, at which the tangent to the parabola is taken.

Plugging in y = 3 into the equation for the parabola $(y - 2)^2 = x - 1$,

$(3 - 2)^2 = x - 1$

$\Rightarrow 1 = x - 1$

$\Rightarrow x = 2$

So, Point 1 is (2, 3)

For point 2:-

We need to find the point of intersection of the parabola and the x – axis.

We know that ordinate at this point is 0.

Plugging in y = 0 into the equation for the parabola $(y - 2)^2 = x - 1$,

$(0 - 2)^2 = x - 1$

$\Rightarrow 4 = x - 1$

$\Rightarrow x = 5$

So, Point 2 is (5, 0)

For point 3:-

Tangent at any point (x_1, y_1) for a curve is –

$y - y_1 = \frac{dy}{dx}(x - x_1)$

For parabola $(y - 2)^2 = x - 1$, differentiating both sides of the equation –

$2(y - 2)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2(y-2)}$

So, slope at point 1, i.e., (2, 3) is –

$$\frac{dy}{dx_{2,3}} = \frac{1}{2(3-2)} = \frac{1}{2}$$

So, equation of tangent at (2, 3) is –

y – 3 = � (x – 2)

⇒ 2y – 6 = x – 2

⇒ x – 2y + 4 = 0

The tangent intersects the x – axis at point 3. At this point, ordinate is 0.

So, plugging y = 0 in the equation of the tangent x – 2y + 4 = 0, -

x – 2 �0 + 4 = 0

⇒ x = -4

So, point 3 is (-4, 0).

Now, that we have the 3 points, let’s figure out how to compute the area required.

The area we need can be divided into 3 sections –

i. x = -4 to x = 1

Here, area required = area enclosed by the tangent and the x – axis

ii. x = 1 to x = 2

Here, area required = (area enclosed by the tangent and the x – axis) – (area enclosed within the parabola)

iii. x = 2 to x = 5

Here, area required = area enclosed by the parabola and the x – axis

The parabola is $(y – 2)^2 = x – 1$

Solving for y,

y = 2 ± √(x – 1)

So, area A we need is –

$$A=\left[\int_{x=-4}^{x=1}\frac{x+4}{2}dx\right]+\left[\int_{x=1}^{x=2}\frac{x+4}{2}dx-\int_{x=1}^{x=2}\left(2+\sqrt{x-1}\right)-\left(2-\sqrt{x-1}\right)dx\right]$$

$$+\left[\int_{x=2}^{x=5}2-\sqrt{x-1}\,dx\right]$$

$$=\frac{1}{4}[(x+4)^2]_{x=-4}^{x=1}+\frac{1}{4}[(x+4)^2]_{x=1}^{x=2}-\frac{4}{3}\left[(x-1)^{\frac{3}{2}}\right]_{x=1}^{x=2}+2[x]_{x=2}^{x=5}-\frac{2}{3}\left[(x-1)^{\frac{3}{2}}\right]_{x=2}^{x=5}$$

$$=\frac{1}{4}[25-0]+\frac{1}{4}[36-25]-\frac{4}{3}[1-0]+2[5-2]-\frac{2}{3}[8-1]$$

$$=9+6-\frac{4}{3}-\frac{14}{3}$$

$$=9+6-\frac{18}{3}$$

$$=9+6-6$$

$= 9$ (Ans)

10. Question

The area bounded by the curves y = sin x between the ordinates x = 0, x = π and the x-axis is

A. 2 sq. units

B. 4 sq. units

C. 3 sq. units

D. 1 sq. units

Answer

This is as simple as –

A = $\int_{x=0}^{x=\pi} \sin x \, dx$

= $[-\cos x]_{x=0}^{x=\pi}$

= [-(-1) + 1]

= 2 (Ans)

11. Question

The area bounded by the parabola $y^2 = 4ax$ and $x^2 = 4ay$ is

A. $\frac{8a^3}{3}$

B. $\frac{16a^2}{3}$

C. $\frac{32\,a^2}{3}$

D. $\frac{64\,a^2}{3}$

Answer

This problem is the generalized form of question 2.

Let’s proceed to solve it similarly –

We need the bounds, i.e., where the area starts and where it ends. At the points of intersection, both the equations are satisfied.

This means that at points of intersection

$y^2 = 4ax$

$\Rightarrow \left(\frac{x^2}{4a}\right)^2 = 4ax$ (from the other equation)

Let us solve this.

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$x^4 = 64a^3x$

$\Rightarrow x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0$ or $x^3 = 64a^3$, i.e, x = 4a

So the points of intersection are (0,0) and (4a,4a)

Now, let’s compute the area.

If we integrate w.r.t x, we’ll have to integrate the space between the two curves from x = 0 to x =4a.

i.e.,

$$A = \int_{x=0}^{x=4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \left[\frac{4}{3a}(ax)^{\frac{3}{2}} - \frac{x^3}{12a}\right]_{x=0}^{x=4a}$$

$= \left[\frac{4}{3a}\cdot\left((4a^2)^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) - \left(\frac{(4a)^3}{12a} - \frac{0^3}{12a}\right)\right]$

$= \left[\frac{4}{3}\cdot 8a^2 - \frac{16a^2}{3}\right]$

$= \left[\frac{32a^2}{3} - \frac{16a^2}{3}\right] = \frac{16a^2}{3}$ (Ans)

12. Question

The area bounded by the curve $y = x^4 - 2x^3 + x^2 + 3$ with x-axis and ordinates corresponding to the minima of y is

A. 1

B. 91/30

C. 30/9

D. 4

Answer

$y = x^4 - 2x^3 + x^2 + 3$

$\Rightarrow y' = 4x^3 - 6x^2 + 2x$

At extrema of y, y' = 0

i.e., $4x^3 - 6x^2 + 2x = 0$

or, $2x^3 - 3x^2 + x = 0$

$\Rightarrow x(2x - 1)(x - 1) = 0$

i.e., x = 0, x = 1/2 and x = 1 correspond to extrema of y

Now, $y'' = 12x^2 - 12x + 2$

$y''_0 = 2 > 0 \Rightarrow x = 0$ corresponds to a minima

$y''_{1/2} = -1 < 0 \Rightarrow x = 1/2$ corresponds to a maxima

$y''_1 = 2 > 0 \Rightarrow x = 1$ corresponds to a minima

So, x = 0 and x = 1 are the ordinates we need.

So, the area A is –

$A = \int_{x=0}^{x=1}(x^4 - 2x^3 + x^2 + 3)\,dx$

$= \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} + 3x\right]_{x=0}^{x=1}$

$= \frac{1}{5} - \frac{1}{2} + \frac{1}{3} + 3$

$= \frac{6 - 15 + 10 + 90}{3}$

$= \frac{91}{30}$ (Ans)

13. Question

The area bounded by the parabola $y^2 = 4ax$, latus rectum and x-axis is

A. 0

B. 4/3 a^2

C. 2/3 a^2

D. $a^2/3$

Answer

The area A is –

A = $\int_{x=0}^{x=a} \sqrt{4ax}\, dx$

$= \frac{2}{3 \cdot 4a}\left[(4ax)^{\frac{3}{2}}\right]_{x=0}^{x=a}$

$= \frac{4}{3}a^2$ (Ans)

14. Question

The area of the region $\{(x, y): x^2 + y^2 \le 1 \le x + y\}$ is

A. π/5

B. π/4

C. π/2 – 1/2

D. $\pi^2/2$

Answer

We need to determine what region is being talked about.

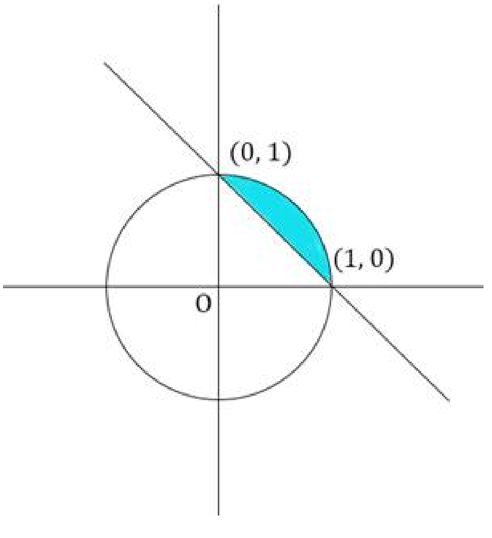
$x^2 + y^2 = 1$ is a circle with centre at origin and unit radius.

So, $x^2 + y^2 \le 1$ represents the region inside that circle.

x + y = 1 is a line that intersects the 2 axes at unit distances from the origin, in the positive direction.

So x + y ≥ 1 represents all the points, i.e., the region above the line x + y = 1.

So, the region $\{(x, y): x^2 + y^2 \le 1 \le x + y\}$ is –



For each point (x, y) on circle in first quadrant, $y = \sqrt{(1 - x^2)}$

For each point (x, y) on line, y = 1 – x

So, area A of the region described is –

A = $\int_{x=0}^{x=1} \left\{\sqrt{1 - x^2} - (1 - x)\right\} dx$

$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - x + \frac{x^2}{2}\right]_{x=0}^{x=1}$

= π/4 – 1/2 (Ans)

15. Question

The area common to the parabola y = $2x^2$ and y = $x^2 + 4$ is

A. 2/3 sq. units

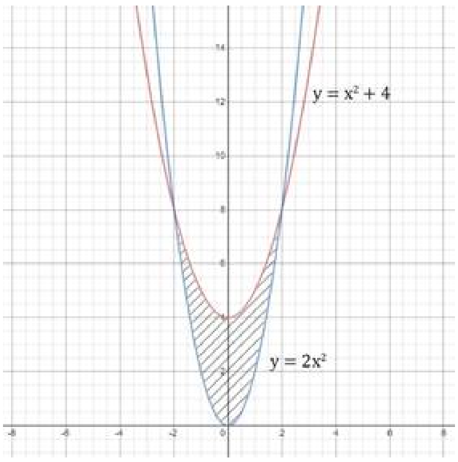
B. 3/2 sq. units

C. 32/3 sq. units

D. 3/32 sq. units

Answer

The area we need looks like –



We need to find the points of intersection to set the bounds of integration.

At points of intersection, y = $x^2 + 4 = 2x^2$

or, $x^2 - 4 = 0$

⇒ (x + 2)(x – 2) = 0

⇒ x = -2, 2

⇒ The points of intersection are (-2, 8) and (2, 8), which is also evident from the graph.

So, the area A of the shaded region is –

A = $\int_{x=-2}^{x=2}\{(x^2+4) - 2x^2\}dx$

$= \int_{x=-2}^{x=2}(4-x^2)\,dx$

Since 4 – x^2 is an even function,

A = $2\int_{x=0}^{x=2}(4-x^2)\,dx$

$= 2\left[4x - \frac{x^3}{3}\right]_{x=0}^{x=2}$

$= 2\left[8 - \frac{8}{3}\right]$

= 32/3 (Ans)

16. Question

The area of the region bounded by the parabola y = $x^2 + 1$ and the straight line x + y = 3 is given by

A. 45/7

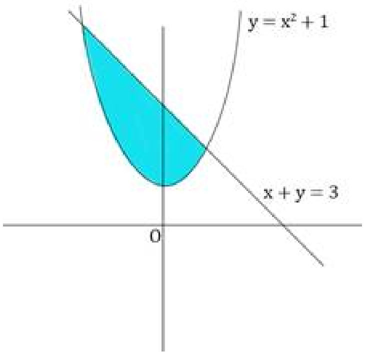
B. 25/4

C. π/18

D. 9/2

Answer

The situation looks like this –



At the intersection points, y = 3 – x = x^2 + 1

Or, $x^2 + x - 2 = 0$

⇒ (x + 2)(x – 1) = 0

⇒ x = -2, 1 → these are our bounds

So, area A enclosed is –

A = $\int_{x=-2}^{x=1}\{(3-x)-(x^2+1)\}dx$

= $\int_{x=-2}^{x=1}\{2-x-x^2\}dx$

= $\left[2x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{x=-2}^{x=1}$

= 2 – 1/2 - 1/3 + 4 + 2 – 8/3

= 8 – 9/3 – 1/2

= 5 – 1/2

= 9/2 (Ans)

17. Question

The ratio of the areas between the curves y = cosx and y = cos 2x and x-axis from x = 0 to x = π/3 is

A. 1 : 2

B. 2 : 1

C. $\sqrt{3} : 1$

D. none of these

Answer

Let us call the corresponding areas A_1 and A_2.

$A_1 = \int_{x=0}^{x=\frac{\pi}{3}} \cos x \, dx$

= $[\sin x]_{x=0}^{x=\frac{\pi}{3}}$

$= \sqrt{3}/2$

$A_2 = \int_{x=0}^{x=\frac{\pi}{3}} \cos 2x \, dx$

$= \frac{1}{2}[\sin 2x]_{x=0}^{x=\frac{\pi}{3}}$

$= \sqrt{3}/4$

$\therefore A_1 : A_2 = 2 : 1$ (Ans)

18. Question

The area between x-axis and curve y = cos x when $0 \le x \le 2\pi$ is

A. 0

B. 2

C. 3

D. 4

Answer

Let area be A.

So, area A is –

$A = \int_{x=0}^{x=2\pi} |\cos x| \, dx$

Now, cos x is positive from x = 0 to x = π/2 and from x = 3π/2 to x = 2π and negative from x = π/2 to x = 3π/2.

So, $A = \int_{x=0}^{x=\frac{\pi}{2}} \cos x \, dx + \int_{x=\frac{\pi}{2}}^{x=\frac{3\pi}{2}} (-\cos x) \, dx + \int_{x=\frac{3\pi}{2}}^{x=2\pi} \cos x \, dx$

$= [\sin x]_{x=0}^{x=\frac{\pi}{2}} + [-\sin x]_{x=\frac{\pi}{2}}^{x=\frac{3\pi}{2}} + [\sin x]_{x=\frac{3\pi}{2}}^{x=2\pi}$

= [1 – 0] + [-(-1) + 1] + [0 – (-1)]

= 1 + 1 + 1 + 1

= 4 (Ans)

19. Question

Area bounded by parabola $y^2 = x$ and straight line 2y = x is

A. 43

B. 1

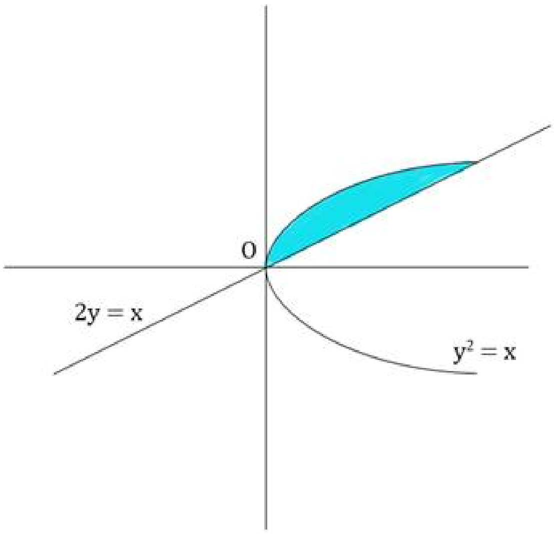
C. 23

D. 13

Answer

Construction:

The situation looks like this –



At the intersection points, x = 2y = y^2

or, y^2 – 2y = 0

⇒ y(y – 2) = 0

⇒ y = 0 or y = 2

So, intersection points are (0, 0) and (4, 2)

Hence, the area A enclosed is –

A = $\int_{y=0}^{y=2}(2y - y^2)\,dy$

= $\left[y^2 - \frac{y^3}{3}\right]_{y=0}^{y=2}$

= [4 – 8/3]

= 4/3 (Ans)

20. Question

The area bounded by the curve y = 4x – x^2 and the x-axis is

A. $\frac{30}{7}$ sq.units

B. $\frac{31}{7}$ sq.units

C. $\frac{32}{3}$ sq.units

D. $\frac{34}{3}$ sq.units

Answer

y = 4x – x^2

This is a parabola with negative co-efficient of x^2, i.e., it's a downward parabola.

So, area A enclosed is the area of the peak of the parabola above the x – axis.

We need to find the bounds of this peak.

Now, at the point where the peak starts/ends, y = 0, i.e., $4x - x^2 = 0$

⇒ $x(4 - x) = 0$

⇒ $x = 0$ or $x = 4$

∴ A = $\int_{x=0}^{x=4} 4x - x^2\, dx$

= $\left[2x^2 - \frac{x^3}{3}\right]_{x=0}^{x=4}$

= [32 – 64/3]

= 32/3 sq. units (Ans)

21. Question

Area enclosed between the curve $y^2 (2a - x) = x^3$ and the line x = 2a above x-axis is

A. πa^2

B. $\frac{3}{2}\pi a^2$

C. $2\pi a^2$

D. $3\pi a^2$

Answer

$y^2(2a - x) = x^3$

⇒ $y^2 = \frac{x^3}{2a-x}$

⇒ $y = \pm\sqrt{\frac{x^3}{2a-x}}$

Since we are concerned with the area above the x – axis, we'll be considering the positive root.

We can see that at x = 0, y = 0.

So,

A = $\int_{x=0}^{x=2a} \sqrt{\frac{x^3}{2a-x}}\, dx$

Putting √x = u

We get du = (1/2√x)dx

or, dx = 2u du

So,

A = $2\int_{u=0}^{u=\sqrt{2a}} \frac{u^4}{\sqrt{2a-u^2}}\, du$

Putting u = √2√a sin t

We get du = √2√a cos t dt

A = $2\int_{t=0}^{t=\frac{\pi}{2}} \frac{4a^2 \sin^4 t}{\sqrt{2a-2a\sin^2 t}} \cdot \sqrt{2a}\cos t\, dt$

$= 8a^2 \int_{t=0}^{t=\frac{\pi}{2}} \frac{\sin^4 t \cos t}{\sqrt{1-\sin^2 t}}\, dt$

$= 8a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^4 t\, dt$

$= 8a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^2 t \cdot \sin^2 t\, dt$

$= 8a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^2 t\,(1 - \cos^2 t)\, dt$

$= 8a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^2 t\, dt - 8a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^2 t \cos^2 t\, dt$

$= 4a^2 \int_{t=0}^{t=\frac{\pi}{2}} (1 - \cos 2t)\, dt - 2a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^2 2t\, dt$

$= 4a^2 \left[t - \frac{1}{2}\sin 2t\right]_{t=0}^{t=\frac{\pi}{2}} - a^2 \int_{t=0}^{t=\frac{\pi}{2}} (1 - \cos 4t)\, dt$

$= 4a^2 \left[t - \frac{1}{2}\sin 2t\right]_{t=0}^{t=\frac{\pi}{2}} - a^2 \left[t - \frac{1}{4}\sin 4t\right]_{t=0}^{t=\frac{\pi}{2}}$

$= 4a^2(\pi/2 - 0 - 0 + 0) - a^2(\pi/2 - 0 - 0 + 0)$

$= \frac{3}{2}\pi a^2$ (Ans)

22. Question

The area of the region (in square units) bounded by the curve $x^2 = 4y$, line x = 2 and x-axis is

A. 1

B. 2/3

C. 4/3

D. 8/3

Answer

$x^2 = 4y$, at y = 0, x = 0

So, our bounds are x = 0 and x = 2

Area A enclosed is –

$A = \int_{x=0}^{x=2} y\, dx$

$= \int_{x=0}^{x=2} \frac{x^2}{4}\, dx$

$= \left[\frac{x^3}{12}\right]_{x=0}^{x=2}$

= 8/12, i.e., 2/3 (Ans)

23. Question

The area bounded by the curve y = f(x), x-axis, and the ordinates x = 1 and x = b is (b – 1) sin (3b + 4). Then, f (x) is

A. (x – 1) cos (3x + 4)

B. sin (3x + 4)

C. sin (3x + 4) +3 (x – 1) cos (3x + 4)

D. none of these

Answer

So, the area enclosed from x = 1 to x = x (say) is (x – 1) sin (3x + 4)

⇒ A = ∫f(x) dx = F(x) = (x – 1) sin (3x + 4)

⇒ f(x) = F'(x) = sin (3x + 4) + 3(x – 1) cos (3x + 4) (Using u-v rule of differentiation)

(Ans)

24. Question

The area bounded by the curve $y^2 = 8x$ and $x^2 = 8y$ is

A. $\frac{16}{3}$ sq. units

B. $\frac{3}{16}$ sq. units

C. $\frac{14}{3}$ sq. units

D. $\frac{3}{14}$ sq. units

Answer

This problem is similar to Problem 2. So we'll solve it in a similar way.

At points of intersection,

$y^2 = 8x$

$\Rightarrow \left(\frac{x^2}{8}\right)^2 = 8x$ (from the other equation)

Let us solve this.

$$\left(\frac{x^2}{8}\right)^2 = 8x$$

$x^4 = 512x$

$\Rightarrow x(x^3 - 512) = 0$

$\Rightarrow x = 0$ or $x^3 = 512$, i.e, $x = 8$

So the points of intersection are (0,0) and (8,8)

Now, let's compute the area.

If we integrate w.r.t x, we'll have to integrate the space between the two curves from x = 0 to x = 8.

i.e.,

$$A = \int_{x=0}^{x=8} \left(2\sqrt{2x} - \frac{x^2}{8}\right) dx$$

$$= \left[\frac{2}{3}(2x)^{\frac{3}{2}} - \frac{x^3}{24}\right]_{x=0}^{x=8}$$

$$= \left[\frac{2}{3} \cdot \left(16^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) - \left(\frac{8^3}{24} - \frac{0^3}{24}\right)\right]$$

$$= \left[\frac{2}{3} \cdot 64 - \frac{64}{3}\right]$$

$= \frac{64}{3}$ (Ans)

25. Question

The area bounded by the parabola $y^2 = 8x$, the x-axis and the latusrectum is

A. 16/3

B. 23/3

C. 32/3

D. $\frac{16\sqrt{2}}{3}$

Answer

Latus rectum of parabola $y^2 = 4ax$ is x = a

So, latus rectum of parabola $y^2 = 8x$ is x = 2

Therefore, area A enclosed is –

A = $\int_{x=0}^{x=2} \sqrt{8x}\, dx$

$= \frac{2}{3} \cdot \frac{1}{8} \left[(8x)^{\frac{3}{2}}\right]_{x=0}^{x=2}$

$= \frac{2}{3} \cdot \frac{1}{8} [64 - 0]$

= 16/3 (Ans)

26. Question

Area bounded by the curve $y = x^3$, the x-axis and the ordinates x = –2 and x = 1 is

A. –9

B. $\frac{-15}{4}$

C. $\frac{15}{4}$

D. $\frac{17}{4}$

Answer

The area A enclosed is –

A = $\int_{x=-2}^{x=1} |x^3|\, dx$

$= \int_{x=-2}^{x=0} (-x^3)\, dx + \int_{x=0}^{x=1} x^3\, dx$

$= \left[-\frac{x^4}{4}\right]_{x=-2}^{x=0} + \left[\frac{x^4}{4}\right]_{x=0}^{x=1}$

= 16/4 + 1/4

= 17/4 (Ans)

27. Question

The area bounded by the curve y = x |x| and the ordinates x = –1 and x = 1 is given by

A. 0

B. 1/3

C. 2/3

D. 4/3

Answer

The area A enclosed is –

A = $\int_{x=-1}^{x=1} |x|x||\, dx$

Now, y = x|x| = $\begin{cases} -x^2, for\ x < 0 \\ x^2, for\ x \geq 0 \end{cases}$

So, |y| = |x|x|| = x^2

∴ A = $\int_{x=-1}^{x=1} x^2\,dx$

= $\left[\frac{x^3}{3}\right]_{x=-1}^{x=1}$

= 1/3 – (-1/3)

= 2/3 (Ans)

28. Question

The area bounded by the y-axis, y = cos x and y = sin x when 0 ≤ x ≤ π/2 is

A. $2(\sqrt{2}-1)$

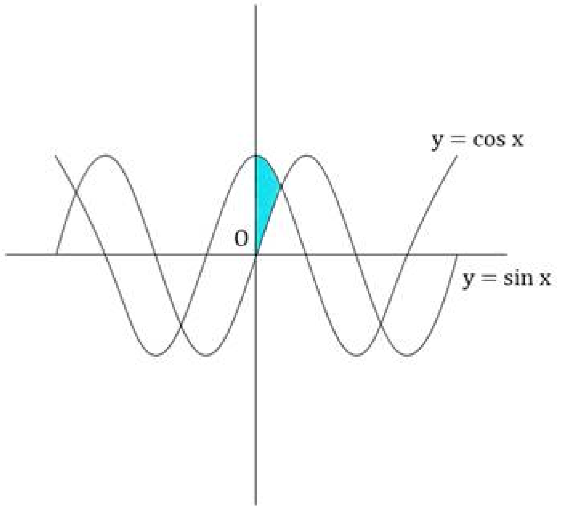
B. $\sqrt{2}-1$

C. $\sqrt{2}+1$

D. $\sqrt{2}$

Answer

The area we want is –



We'll integrate w.r.t y, since area is enclosed by curves and y – axis.

Intersection point is at x = π/4, i.e., y = 1/√2

So, area A enclosed is –

A = $\int_{y=0}^{y=\frac{1}{\sqrt{2}}} \sin^{-1} y\,dy + \int_{y=\frac{1}{\sqrt{2}}}^{y=1} \cos^{-1} y\,dy$

Using Integration by parts –

A = $\left[\sin^{-1} y \int dy - \int \frac{y}{\sqrt{1-y^2}} dy\right]_{y=0}^{y=\frac{1}{\sqrt{2}}} + \left[\cos^{-1} y \int dy - \int -\frac{y}{\sqrt{1-y^2}} dy\right]_{y=\frac{1}{\sqrt{2}}}^{y=1}$

Putting u = 1 – y^2

We get du = -2y dy

A = $\left[y\sin^{-1}y + \frac{1}{2}\int\left(\frac{1}{\sqrt{u}}\right)du\right]_{y=0}^{y=\frac{1}{\sqrt{2}}} + \left[y\cos^{-1}y - \frac{1}{2}\int\left(\frac{1}{\sqrt{u}}\right)du\right]_{y=\frac{1}{\sqrt{2}}}^{y=1}$

$= \left[y\sin^{-1}y + \sqrt{u}\right]_{y=0}^{y=\frac{1}{\sqrt{2}}} + \left[y\cos^{-1}y - \sqrt{u}\right]_{y=\frac{1}{\sqrt{2}}}^{y=1}$

$= \left[y\sin^{-1}y + \sqrt{1-y^2}\right]_{y=0}^{y=\frac{1}{\sqrt{2}}} + \left[y\cos^{-1}y - \sqrt{1-y^2}\right]_{y=\frac{1}{\sqrt{2}}}^{y=1}$

$= \pi/4\sqrt{2} + 1/\sqrt{2} - 0 - 1 + 0 - 0 - \pi/4\sqrt{2} + 1/\sqrt{2}$

$= \sqrt{2} - 1$ (Ans)

29. Question

The area of the circle $x^2 + y^2 = 16$ enterior to the parabola $y^2 = 6x$ is

A. $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$

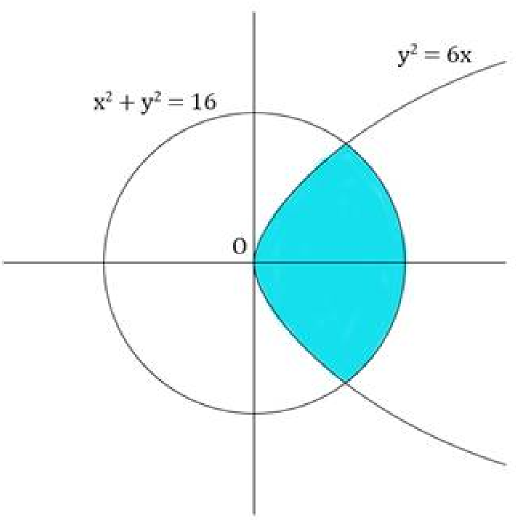
B. $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$

C. $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$

D. $\frac{4}{3}\left(8\pi + \sqrt{3}\right)$

Answer

The area we want is –



At intersection points, $y^2 = 6x = 16 - x^2$

Or, $x^2 + 6x - 16 = 0$

i.e., $(x + 8)(x - 2) = 0$

i.e., $x = -8, 2$

Now $x = -8 \Rightarrow y^2 = 16 - (-8)^2 = 16 - 64 = -48 < 0$, which is not possible for $y \in \mathbb{R}$

So, $x = 2$

and $y^2 = 16 - 2^2$

$= 16 - 4$

= 12

Or, y = √12 = ± 2√3

So, our bounds are y = -2√3 to y = 2√3

The area A enclosed is –

A = $\int_{y=-2\sqrt{3}}^{y=2\sqrt{3}}\left(\sqrt{16-y^2}-\frac{y^2}{6}\right)dy$

Since both the functions are symmetrical about the x – axis,

A = $2\int_{y=0}^{y=2\sqrt{3}}\left(\sqrt{16-y^2}-\frac{y^2}{6}\right)dy$

$= 2\left[\frac{y}{2}\sqrt{16-y^2}+8\sin^{-1}\frac{y}{4}-\frac{y^3}{18}\right]_{y=0}^{y=2\sqrt{3}}$

$= 2[2\sqrt{3}+\frac{8\pi}{3}-\frac{4\sqrt{3}}{3}-0-0+0]$

$= 2\left[\frac{8\pi}{3}+\frac{2\sqrt{3}}{3}\right]$

$= \frac{4}{3}[4\pi+\sqrt{3}]$

(Ans)

30. Question

Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line x + y = 2 is

A. 2(π –2)

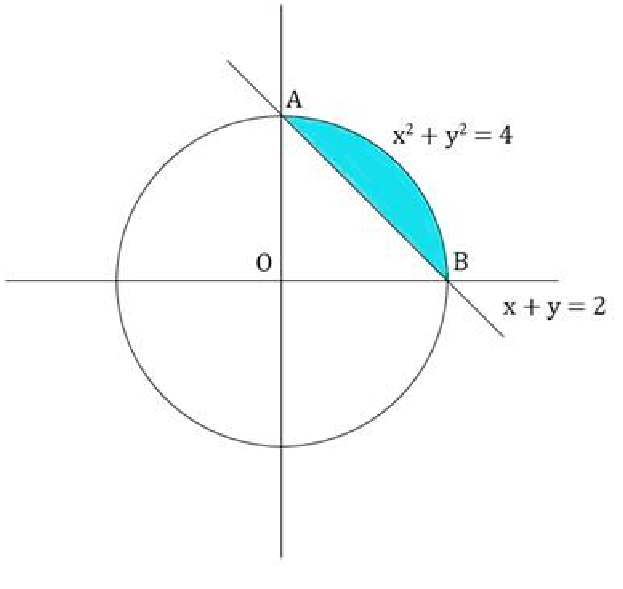
B. π –2

C. 2π –1

D. 2(π + 2)

Answer

The area we need is –



This area A = area of quadrant OAB – area of triangle OAB

$= \pi r^2/4$ – � x base x height

$= \pi \times 2^2/4$ – � x 2 x 2

$= \pi - 2$ (Ans)

31. Question

Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

A. 2/3

B. 1/3

C. 1/4

D. 3/4

Answer

At the points of intersection,

$x = y^2/4 = y/2$

Or, $y^2 = 2y$

i.e., $y^2 - 2y = 0$

i.e., $y(y - 2) = 0$

i.e., $y = 0, 2$

So, area A enclosed is –

$$A = \int_{y=0}^{y=2} \left(\frac{y}{2} - \frac{y^2}{4}\right) dy$$

$$= \left[\frac{y^2}{4} - \frac{y^3}{12}\right]_{y=0}^{y=2}$$

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

(Ans)

32. Question

Area lying in first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$, is

A. π

B. π/2

C. π/3

D. π/4

Answer

The part of the circle $x^2 + y^2 = 4$ in between $x = 0$ and $x = 2$ is the semicircle to the right of the y – axis.

And the part of this semicircle in the first quadrant is a quadrant of the circle.

So, area A of the portion is basically the area of a quadrant of the circle.

$\therefore A = \pi r^2/4$

$= \pi \times 2^2/4$

= π (Ans)

33. Question

Area of the region bounded by the cure $y^2 = 4x$, y-axis and the line y = 3, is

A. 2

B. 9/4

C. 9/3

D. 9/2

Answer

The area A is –

A = $\int_{y=0}^{y=3} \frac{y^2}{4} dy$

= $\left[\frac{y^3}{12}\right]_{y=0}^{y=3}$

= 9/4 (Ans)